

Instantons on multi-Taub-NUT Spaces I: Asymptotic Form and Index Theorem

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Abstract

We study finite action anti-self-dual Yang-Mills connections on the multi-Taub-NUT space. We establish the curvature and the harmonic spinors decay rates and compute the index of the associated Dirac operator.

This is the first in a series of papers proving the completeness of the bow construction of instantons on multi-Taub-NUT spaces and exploring it in detail.

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1 Introduction

This paper establishes core analytic results needed for proving the completeness of the bow construction [Che11] of instantons on Asymptotically Locally Flat (ALF) spaces. In general, for an oriented Riemannian four-manifold M , we call a connection A on a rank n Hermitian bundle $E \rightarrow M$ an *instanton* on M if it has square integrable, anti-self-dual curvature F_A . Our focus is on the case when M is a prototypical ALF space: the multi-Taub-NUT space. The bow construction, similar to the AHDM-Nahm construction [AHDM78, Nah83, Nah84], relates instantons on ALF spaces to solutions of a system of nonlinear ordinary differential equations on a collection of line segments and additional linear data on their boundary. This data is conveniently organized in terms of a bow. In comparison, the equivariant version of the ADHM construction, studied by Kronheimer and Nakajima [KN90], relates instantons on Asymptotically Locally Euclidean (ALE) spaces to solutions of algebraic equations originating from a quiver. In both cases, the rank of the bow or quiver system is determined by the dimension of the space of L^2 harmonic spinors on the ALF or ALE space, twisted by an instanton connection. In this paper we compute this dimension with an index theorem, as a first step in proving the above correspondence. Establishing

the asymptotic form of the connection is crucial for the index calculation as well as for the asymptotic form of these harmonic spinors.

In order to establish the asymptotic form of the connection, we first prove quadratic decay of the curvature. The curvature of the known $U(1)$ instantons [Yui87, BFRM96] on ALF spaces decays quadratically; so, the desired decay rate is sharp. Establishing quadratic decay for ALF spaces is more delicate than proving quartic decay in the ALE case. To see this, consider the ordinary differential equation on $[1, \infty)$:

$$\left(-\frac{d^2}{dr^2} + \frac{p}{r^2}\right)y = 0.$$

The solutions of this equation are $c_1 r^{\frac{1}{2} + \sqrt{p + \frac{1}{4}}} + c_2 r^{\frac{1}{2} - \sqrt{p + \frac{1}{4}}}$, and the decay rate is determined by the magnitude of the quadratically decaying term. The analogous equation for Yang-Mills equations is the Bochner formula

$$\nabla^* \nabla F_A - \epsilon^i \epsilon_j^* R_{ij} F_A - \epsilon^i \epsilon_j^* [F_{ij}, F_A] = 0.$$

Here R denotes the Riemann curvature, which decays cubically for ALF spaces, and therefore will not critically affect our estimates. The $\text{ad}(F_{ij})$ term, however, is analogous to the $\frac{p}{r^2}$ term, with p completely unknown. Hence, the unknown magnitude of the curvature, which we wish to bound, appears to play an important role in establishing bounds. Terms with faster than quadratic decay are negligible in such analyses, making the ALE case significantly simpler. In particular, the Sobolev and Hardy inequalities for ALE spaces are stronger than for ALF spaces. These inequalities, coupled to a Moser iteration argument, readily imply that instantons on ALE spaces decay faster than r^{-q} for some $q > 2$. Once such decay is demonstrated, the ALE decay problem is effectively linearized and readily solved.

As proved in [Min10, CC15] ALF spaces with faster than quadratic curvature decay are asymptotic to the multi-Taub-NUT metric, described below. In this paper, we therefore focus our attention on the case of M a k -centered Taub-NUT (TN_k) space; the general ALF (with faster than quadratic curvature decay) case then follows from essentially the same argument. TN_k is hyperkähler with a triholomorphic isometric circle action, which has k fixed points, $\{\nu_1, \dots, \nu_k\}$. The quotient of TN_k by this circle action is \mathbb{R}^3 , and the quotient map $\pi_k : \text{TN}_k \rightarrow \mathbb{R}^3$ defines a principal circle fibration in the complement of the fixed points:

$$S^1 \rightarrow M \setminus \{\nu_1, \dots, \nu_k\} \xrightarrow{\pi_k} \mathbb{R}^3 \setminus \{\nu_1, \dots, \nu_k\}. \quad (1)$$

This S^1 bundle has Hopf number -1 over any small sphere centered at any fixed point ν_σ . Let ϖ be a connection one-form on this bundle that is metrically dual to an isometry-generating vector field and that has curvature $d\varpi = \pi_k^*(*_3 dV)$, where $V = l + \sum_{\sigma=1}^k \frac{1}{|x - \nu_\sigma|}$. In terms of this one-form, the hyperkähler metric on TN_k has the following form found by Gibbons and Hawking [GH78]:

$$V dx^2 + \frac{\varpi^2}{V}. \quad (2)$$

Our analysis assumes a generic asymptotic holonomy condition: the holonomy of the instanton connection around the Taub-NUT circle fiber is generic as we approach infinity along some fixed direction in the base \mathbb{R}^3 . In other words, we assume that the limiting eigenvalues of the holonomy are distinct. Under these assumptions we prove in Section 3 the following

Theorem 14. Let M be an ALF space and A an instanton connection with generic asymptotic holonomy. Let $p \in M$. Then there exists $C > 0$ so that $|F_A|(x) \leq Cd(x, p)^{-2}$.

We prove in Section 4 that the conjugacy class of the asymptotic holonomy is independent of the direction in which we approach infinity. Thus, outside a compact set, the centralizer of the holonomy around the ALF circle fiber is a Cartan subgroup of the gauge group. In local coordinates $(\vec{x}, \tau) \in U \times S^1$ with $U \subset \mathbb{R}^3 \setminus \{\nu_j\}_{j=1}^k$ an open contractible neighborhood and $\tau \in [0, 4\pi)$ a homogeneous coordinate along the circle fiber, we have $\varpi = d\tau + \pi_k^* \omega$, with ω a one-form¹ on U satisfying $d\omega = *_3 dV$. Using the quadratic decay established in Theorem 14, we prove in Section 4 that the asymptotic form of the instanton connection is that of a direct sum of the known $U(1)$ instantons²:

Theorem 18. There exist local frames for E in which the instanton connection one-form is $A = -i \operatorname{diag}(a_1, a_2, \dots, a_n) + O(\frac{\ln r}{r^2})$, with

$$a_j = \left(\lambda_j + \frac{m_j}{r}\right) \frac{d\tau + \omega}{2V} - \frac{m_j}{2k} \omega, \text{ where } \lambda_j \in \mathbb{R} \text{ and } m_j \in \mathbb{Z} \text{ for } 1 \leq j \leq n.$$

In Section 5 we prove decay estimates for the L^2 harmonic spinors, which will be needed in our subsequent analysis of the bow construction.

Proposition 20 and **Theorem 21.** Let A be an instanton on TN_k with generic asymptotic holonomy and $\psi \in \operatorname{Ker}(D_A^+) \cap L^2$, then $|\psi|$ decays exponentially if the asymptotic holonomy has no invariant vectors, i.e. if $\frac{\lambda_j}{l} \notin \mathbb{Z}$ for all j , and decays quadratically otherwise.

We fix the orientation of TN_k by setting $d\text{Vol} = V d\text{Vol}_{\mathbb{R}^3} \wedge \varpi$. With respect to this orientation, the Kähler forms are self-dual, and the Riemann curvature is anti-self-dual. Under the Clifford action of the volume form the spin bundle S splits as $S = S^+ \oplus S^-$ with S^+ and S^- denoting respectively, the positive and the negative eigenvalue eigen-bundles of $Cl(d\text{Vol})$. Due to hyperkählerity, S^- is trivial. Let D_A^\pm denote the restriction of D_A to sections of $S^\pm \otimes E$. With our orientation and chirality conventions, $\operatorname{Ker}(D_A^-) \cap L^2 = \{0\}$. Hence, in order to compute the dimension of $\operatorname{Ker}(D_A) \cap L^2$ it suffices to compute the L_2 index of D_A^+ . We prove the following index theorem in Section 6:

Theorem 35. The L^2 -index of D_A^+ is

$$\operatorname{ind}_{L^2} D_A^+ = \sum_j \left(\left(\frac{1}{2} - \{\lambda_j/l\} \right) (k \lfloor \lambda_j/l \rfloor - m_j) - \frac{k}{2} \{\lambda_j/l\}^2 \right) + \frac{1}{8\pi^2} \int F \wedge F, \quad (3)$$

where $\lfloor a \rfloor$ denotes the largest integer not greater than a and $\{a\} = a - \lfloor a \rfloor$.

¹ For the sake of brevity, in the rest of the paper we shall use ω to denote $\pi_k^*(\omega)$ as well, thus writing $\varpi = d\tau + \omega$, just as we used V for $\pi_k^*(V)$ in Eq. (2) and ν_σ for $\pi_k^*(\nu_\sigma)$ in (1), hoping this will cause no confusion.

²The fact that all $U(1)$ instantons have this form is proved in [HHM04].

There has been extensive work by numerous authors analyzing the decay rates of the curvature of Yang-Mills and instanton connections. Uhlenbeck [Uhl82, Uhl79] showed that a finite energy Yang-Mills connection on \mathbb{R}^4 has quartic curvature decay. This result has been generalized in many directions, including [FU91], [IN90], [DK90, Sec.4.4.3], and [Råd93]. In particular, Groisser and Parker [GP97] extend the quartic decay to asymptotically flat spaces (as defined in [GP97, Sec.1]), and their proof readily extends to ALE spaces.

The decay rate is sensitive to the asymptotic form of the metric, especially to the asymptotic volume growth. For example, Mochizuki [Moc14] studied the case of doubly periodic instantons, proving $|F| = O(1/r^{1+\epsilon})$. A very closely related case to the one we consider is that of a monopole on \mathbb{R}^3 , for which Jaffe and Taubes [JT80] proved quadratic decay of the curvature.

Our techniques for analyzing the decay of harmonic spinors in the Fredholm case follow closely the work of Agmon [Agm82]. The only novelty in our treatment lies in our modification of the Bochner formula required to treat the non-Fredholm case. The decay of harmonic spinors on spaces with quadratically decaying Green's operator (such as one finds in ALE spaces) was also considered in [Fee01].

A Perspective

Moduli spaces of instantons on multi-Taub-NUT are gaining significance in both mathematics and physics. They play a central role in the Geometric Langlands correspondence for complex surfaces. The original versions of this correspondence [BF10, BF12] focused on instantons on ALE spaces, however, the physics picture [Tan10, Wit10] reveals that instantons on multi-Taub-NUT tell a richer story. Another significance of these instanton moduli spaces is their emergence as Coulomb branches of three-dimensional $N = 4$ supersymmetric gauge theories [SW96, dBHO097, NT16] as well as both the Coulomb and Higgs branches of Seiberg-Witten theories with impurities [GW09, COS11]. A mathematical treatment of these spaces and their relation to bows appeared in [NT16].

These interpretations of the instanton moduli space give precise predictions for the dimension of their L^2 cohomology. See, e.g. [MRB15] for the case of monopole moduli spaces, [Nak94, BF10] for the case of instantons on ALE spaces, and [Wit10, CHZ14, Nak15] for our case of instantons on ALF spaces. It is a challenging problem to verify these by directly computing the L^2 cohomology. In fact, since the direct study of instanton moduli spaces presents numerous analytic challenges, it is desirable to have a simpler realization of the same spaces. The bow construction [Che11] suggests that instantons are in correspondence with bow solutions and that instanton moduli spaces are isomorphic to bow moduli spaces. The bow moduli spaces are much more amenable to computation. For example, their asymptotic metric was found in [Che11] and, for the metric on the moduli space of a low rank bow representation, it was computed explicitly in [Che09]. In this sequence of papers, we set out to prove that, indeed, bow moduli spaces are isometric to the moduli spaces of instantons on multi-Taub-NUT space.

Another significant applications of instantons on multi-Taub-NUT space is in string theory where they deliver an effective description of the Chalmers-Hanany-Witten brane configurations [CH97, HW97]. This relation provides significant information about the instantons themselves, as demonstrated in [Wit09].

2 Yang-Mills Connections

Our preliminary estimates for the curvature decay require only the weaker hypothesis that the connection is smooth, finite action Yang-Mills. Hence for the convenience of the reader, we first work under this weaker hypothesis.

We call a connection A a *Yang-Mills connection* if its curvature satisfies the Yang-Mills equation $d_A^* F_A = 0$. Note, that the Bianchi identity, $d_A F_A = 0$, implies that any connection with anti-self-dual curvature is also a Yang-Mills connection. Let (M, g) be a smooth complete 4-dimensional manifold with bounded geometry, i.e. its injectivity radius is bounded below and the pointwise norm of the Riemann curvature tensor and its covariant derivatives is bounded above. In particular, the volume of a ball of radius R centered at a point $x \in M$, $\text{Vol}(B_R(x))$, is bounded below by an increasing function of R , that is independent of x . The Sobolev embedding theorem holds for such geometries, in particular:

Proposition 1. [Heb99, Sec. 3]. *There exists $C_M > 0$ such that for all $f \in H_1^2(M)$,*

$$\|df\|^2 + \|f\|^2 \geq C_M \|f\|_{L^4}^2. \quad (4)$$

Proposition 2. *Let A be a smooth Yang-Mills connection with L^2 curvature on a Hermitian vector bundle over a Riemannian 4-manifold of bounded geometry. Then $\|F_A\|_{L^\infty} < \infty$.*

We first prove a simple lemma.

Lemma 3. *Let A be a finite action Yang-Mills connection over a Riemannian 4-manifold of bounded geometry. Then $F_A \in L^4$ and $\nabla F_A \in L^2$.*

Proof. Without loss of generality, assume that M is connected. Let³ ε^j denote exterior multiplication on the left by e^j and ε_j^* to denote the adjoint operation - interior multiplication by the metrically dual vector field. With this notation, F_A satisfies the Bochner formula:

$$0 = (d_A + d_A^*)^2 F_A = \nabla^* \nabla F_A - \varepsilon^i \varepsilon_j^* R_{ij} F_A - \varepsilon^i \varepsilon_j^* [F_{ij}, F_A]. \quad (5)$$

Let $\eta \in C_0^\infty(M)$. Taking the L^2 -inner product of the Bochner formula with $\eta^2 F_A$ and integrating by parts yields

$$0 = \|\nabla(\eta F_A)\|^2 - \|d\eta|F_A\|^2 - \langle \varepsilon^i \varepsilon_j^* R_{ij} F_A, \eta^2 F_A \rangle - \langle [F_{ij}, F_{jk}], \eta^2 F_{ik} \rangle. \quad (6)$$

³ In general, for a form ω , $\varepsilon(\omega)$ will denote the exterior multiplication on the left by ω , while $\varepsilon^*(\omega)$ will denote the adjoint operation of the interior product.

Fix $p \in M$ and let χ_L denote the characteristic function for the geodesic ball of radius L about p . Kato's inequality $|\nabla(\eta F_A)| \geq |d\eta F_A|$ and the Sobolev inequality for functions applied to (6) give

$$0 \geq C_M \|\eta F_A\|_{L^4}^2 - \|d\eta F_A\|^2 - (1 + C_R) \|\eta F_A\|^2 - \langle \chi_L [F_{ij}, F_{jk}], \eta^2 F_{ik} \rangle - 4\sqrt{2} \|(1 - \chi_L) F_A\| \|\eta F_A\|_{L^4}^2. \quad (7)$$

Here C_R is a constant depending only on the upper bound for the norm of the Riemann curvature tensor. The assumption that $F_A \in L^2$ implies that we may choose L sufficiently large so that $8\|(1 - \chi_L) F_A\| \leq \frac{C_M}{2}$. Fixing such an L , relation (7) implies

$$\frac{C_M}{2} \|\eta F_A\|_{L^4}^2 \leq \|d\eta F_A\|^2 + (1 + C_R) \|\eta F_A\|^2 + \langle \chi_L [F_{ij}, F_{jk}], \eta^2 F_{ik} \rangle. \quad (8)$$

Taking a sequence of functions $\eta = \eta_n : M \rightarrow [0, 1]$ with $|d\eta_n|$ bounded and $\eta_n \rightarrow 1$ pointwise, reduces this inequality to

$$\frac{C_M}{2} \|F_A\|_{L^4}^2 \leq (1 + C_R) \|F_A\|^2 + \langle \chi_L [F_{ij}, F_{jk}], F_{ik} \rangle. \quad (9)$$

In particular, $F_A \in L^4$. Now in (6) set $\eta = \eta_n$ and send $n \rightarrow \infty$ to deduce $\nabla F_A \in L^2$. \square

We now modify the proof of Lemma 3 to obtain integrability for all p .

Lemma 4. *The curvature of a finite action Yang-Mills connection over a Riemannian 4-manifold of bounded geometry is L^p , for any $p \geq 2$.*

Proof. To pass from L^4 to L^p bounds we choose a different η . For $F_A \in L^2 \cap L^4$ with $\nabla F_A \in L^2$, it suffices that $|\eta| + |d\eta|$ is bounded in order to justify integration by parts in our Bochner arguments.

Consider again the inner product of the Bochner formula (5) with $\eta^2 F_A$. Writing

$$\begin{aligned} \langle \nabla(\eta^2 F_A), \nabla F_A \rangle &= \left\langle \nabla \left(\eta^2 |F_A| \frac{F_A}{|F_A|} \right), \nabla \left(|F_A| \frac{F_A}{|F_A|} \right) \right\rangle \\ &= \langle d(\eta^2 |F_A|), d|F_A| \rangle + \left\| \eta |F_A| \nabla \frac{F_A}{|F_A|} \right\|^2, \end{aligned}$$

the Bochner formula yields

$$\langle d(\eta^2 |F_A|), d|F_A| \rangle \leq \langle \varepsilon^i \varepsilon_j^* R_{ij} \eta F_A, \eta F_A \rangle + \langle [F_{ij}, \eta F_{jk}], \eta F_{ik} \rangle. \quad (10)$$

Consider a sequence of functions

$$\tilde{\eta}(t) = \tilde{\eta}_{n,p}(t) = \min\{t^p, n^p\},$$

and let $\eta = \eta_{n,p} = \tilde{\eta}_{n,p}(|F_A|^2)$. Let χ_n denote the characteristic function of the set $\{x : |F_A|^2(x) \leq n\}$. Then we have $d\eta_{n,p} = 2p\chi_n|F_A|^{2p-1}d|F_A|$ and

$$d(\eta^2|F_A|) = (4p+1)\chi_n|F_A|^{4p}d|F_A| + (1-\chi_n)n^{2p}d|F_A|,$$

and (10) yields

$$\begin{aligned} 0 &\geq (4p+1)\|\chi_n|F_A|^{2p}d|F_A|\|^2 + \|(1-\chi_n)n^{2p}d|F_A|\|^2 \\ &\quad - \langle \varepsilon^i \varepsilon_j^* R_{ij} \eta F_A, \eta F_A \rangle - \langle \varepsilon^i \varepsilon_j^* [F_{ij}, \eta F_A], \eta F_A \rangle \\ &= \frac{(4p+1)}{(2p+1)^2} \|d\eta_{n,p+\frac{1}{2}}\|^2 + \|(1-\chi_n)n^{2p}d|F_A|\|^2 \\ &\quad - \langle \varepsilon^i \varepsilon_j^* R_{ij} \eta_{n,p} F_A, \eta_{n,p} F_A \rangle - \langle [F_{ij}, \eta_{n,p} F_{jk}], \eta_{n,p} F_{ik} \rangle \\ &\geq \frac{(4p+1)}{(2p+1)^2} C_M \|\eta_{n,p+\frac{1}{2}}\|_{L^4}^2 + \|(1-\chi_n)n^{2p}d|F_A|\|_{L^2}^2 \\ &\quad - C_R \|\eta_{n,p} F_A\|_{L^2}^2 - 2\|F_A\|_{L^4} \|\eta_{n,p} F_A\|_{L^{8/3}}^2. \end{aligned}$$

Setting $C = 2\|F_A\|_{L^4}$, we have

$$\frac{(4p+1)}{(2p+1)^2} C_M \|\eta_{n,p+\frac{1}{2}}\|_{L^4}^2 \leq C_R \|\eta_{n,p} F_A\|_{L^2}^2 + C \|\eta_{n,p} F_A\|_{L^{8/3}}^2. \quad (11)$$

Also, since $F_A \in L^4$, $C_R \|\eta_{n,p} F_A\|_{L^2}^2 + C \|\eta_{n,p} F_A\|_{L^{8/3}}^2$ is uniformly bounded as $n \rightarrow \infty$ for all $0 \leq p \leq \frac{1}{4}$. We may therefore take the limit as $n \rightarrow \infty$ in (11) to deduce for $0 \leq p \leq \frac{1}{4}$.

$$\frac{(4p+1)}{(2p+1)^2} C_M \|F_A\|_{L^{8p+4}}^{4p+2} \leq C_R \|F_A\|_{L^{4p+2}}^{4p+2} + C \|F_A\|_{L^{\frac{16p+8}{3}}}^{4p+2}. \quad (12)$$

Choosing $p = \frac{1}{4}$ in this relation yields $F_A \in L^6$, and we may now iterate the argument to deduce that $F_A \in L^p$ for all $p \geq 2$ and that the above inequality (12) holds for all $p \geq 0$. \square

2.1 L^∞ via Moser Iteration

We now prove Proposition 2.

Proof. We apply a standard Moser iteration argument to obtain an L^∞ bound for F_A . To simplify the algebra, we first assume that $|F_A(x)| > 1$ for some $x \in M$. Otherwise the L^∞ bound is immediate. With this added assumption, there exists $C_3 > 0$ independent of p , such that $C_R \|F_A\|_{L^{4p+2}}^{4p+2} + C \|F_A\|_{L^{\frac{16p+8}{3}}}^{4p+2} \leq C_3 C_M \|F_A\|_{L^{\frac{16p+8}{3}}}^{4p+2}$ and (12) becomes for $q = \frac{16p+8}{3}$ and $\gamma = \frac{3}{2}$,

$$\|F_A\|_{L^{\gamma q}} \leq \left[\frac{C_3 (\frac{3q}{8})^2}{(\frac{3q}{4} - 1)} \right]^{\frac{4}{3q}} \|F_A\|_{L^q}. \quad (13)$$

Applying this inequality n times produces

$$\|F_A\|_{L^{\gamma^n q}} \leq \|F_A\|_{L^q} \prod_{j=0}^{n-1} \left[\frac{C_3 \left(\frac{3q\gamma^j}{8} \right)^2}{\left(\frac{3q\gamma^j}{4} - 1 \right)} \right]^{\frac{4}{3q\gamma^j}}. \quad (14)$$

The product on the right hand side remains bounded as $n \rightarrow \infty$, and, with $q = 4$ one obtains

$$\|F_A\|_{L^\infty} \leq \|F_A\|_{L^4} \prod_{j=0}^{\infty} \left[\frac{C_3 \left(\frac{3q\gamma^j}{8} \right)^2}{\left(\frac{3q\gamma^j}{4} - 1 \right)} \right]^{\frac{4}{3q\gamma^j}}. \quad (15)$$

The infinite product on the right hand side is convergent. Combined with relation (9), this result implies that the L^2 norm (plus data on a compact set) controls the L^∞ norm. \square

2.2 ∇F_A and Curvature Decay

Lemma 5. *Let A be a smooth finite action Yang-Mills connection on a vector bundle over a Riemannian 4-manifold of bounded geometry. Then $\nabla F_A \in L^\infty$.*

Proof. The proof follows from the argument of the preceding proposition as soon as we establish a Bochner formula for ∇F_A . In a local orthonormal frame, we have

$$[\nabla^* \nabla, \nabla] F_A = 2\omega^i \otimes [F_{ij}, \nabla_j F_A] + 2R_{ijpq} F_{pt;j} \omega^i \otimes \omega^q \wedge \omega^t. \quad (16)$$

Hence

$$(\nabla^* \nabla) \nabla F_A = \nabla(\epsilon^i \epsilon_j^* (R_{ij} F_A + [F_{ij}, F_A])) + 2\omega^i \otimes ([F_{ij}, \nabla_j F_A] + 2R_{ij} F_{A;j}). \quad (17)$$

We may now repeat the argument of the preceding theorem, with Eq. (17) playing the role of the Bochner formula (5), to deduce $\nabla F_A \in L^\infty$. \square

Lemma 6. *Let A be a smooth finite action Yang-Mills connection on a vector bundle over a Riemannian 4-manifold M of bounded geometry and choose some point $o \in M$. Then $|F_A(x)| \rightarrow 0$ as $d(x, o) \rightarrow \infty$.*

Proof. Let $R > 0$ be fixed and let $r = d(x, o)$. This result can be proved by a standard *local* Moser iteration argument, restricting to a ball $B_{2R}(x)$ and bounding $\|F_A\|_{L^\infty(B_R(x))}$ in terms of $\|F_A\|_{L^2(B_{2R}(x))}$, and using the finiteness of the Yang-Mills action to see that $\lim_{r \rightarrow \infty} \|F_A\|_{L^2(B_{2R}(x))} = 0$. The pointwise gradient estimate may be employed to give an alternative proof. The mean value theorem and the estimate $\|\nabla F_A\|_{L^\infty} < C$ imply that, if $y \in B_{\frac{|F_A(x)|}{2C}}(x)$, then $|F_A(y)| \geq \frac{1}{2}|F_A(x)|$. Hence

$$\int_{B_{\frac{|F_A(x)|}{2C}}(x)} |F_A|^2 dv \geq \frac{1}{4} |F_A(x)|^2 \text{Vol} \left(B_{\frac{|F_A(x)|}{2C}}(x) \right). \quad (18)$$

Once again, the finiteness of the action implies $\lim_{r(x) \rightarrow \infty} |F_A(x)| = 0$.

Indeed, if the pointwise norm $|F_A(x)|$ does not tend to zero as $r(x) \rightarrow \infty$, then there is a sequence of points x_n such that the balls $B_{\frac{|F_A(x_n)|}{2C}}(x_n)$ are disjoint and $|F_A(x_n)| > \epsilon > 0$. Bounded geometry and non-compactness of the underlying space imply $\text{Vol}\left(B_{\frac{|F_A(x_n)|}{2C}}(x_n)\right) > Cf(\epsilon)$, with $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ increasing function. Thus, the contribution of each ball to $\|F_A\|_{L^2}$ is bounded below by a constant $C\epsilon f(\epsilon)$. Since these balls are disjoint, we have a contradiction with F_A being square integrable. \square

3 Instanton Connections

In the previous section we proved that, on a complete four-manifold of bounded geometry, Yang-Mills connections with square integrable curvature have curvature vanishing at infinity with bounded covariant derivative. Now we specialize to anti-self-dual-connections with square integrable curvature on TN_k , and we impose the generic asymptotic holonomy assumption. We first prove in Section 3.1 bounds for the variation of the conjugacy class of the holonomy. In Section 3.2 we show curvature decays at least as fast as $r^{-3/2}$. The next Section 3.3 sharpens this result to quadratic curvature decay.

3.1 Holonomy

We recall that the multi-Taub-NUT metric (1) admits an isometric S^1 action with k fixed points $\{\nu_1, \dots, \nu_k\}$, and the quotient of TN_k by the S^1 action is \mathbb{R}^3 . Let π_k again denote the projection to the \mathbb{R}^3 quotient. Choose a simple unit speed curve c in $\mathbb{R}^3 \setminus \{\nu_1, \dots, \nu_k\}$ and a local trivialization $S^1 \times c$ of the S^1 bundle $\pi_k^{-1}(c)$. Given any frame for E along $\pi_k^{-1}(c(0))$, we extend it to the cylinder $\pi_k^{-1}(c)$ by requiring it to be covariant constant along $\{\tau\} \times c$ for each value τ .

The connection matrix in this frame, for the connection pulled back to the associated cylinder, satisfies

$$A(\tau, s) = A(\tau, 0) + \int_0^s F(c'(u), \cdot)(\tau, u) du. \quad (19)$$

Since c is unit speed, $|F(c', \partial_\tau)| \leq |F|$, and

$$|A(\tau, s) - A(\tau, 0)| \leq |s| \sup_{t \in [0, s]} |F(\tau, c(t))|. \quad (20)$$

Now let $\{v_0^j\}_{j=1}^n$ be a $\frac{\partial}{\partial \tau} + A(\tau, 0)$ covariant constant frame along $\pi_k^{-1}(c(0))$. As before, extend it to a frame $\{v_s^j\}_j$ along the cylinder. Let $\{w^i = w_j^i v_s^j\}_i$ be a $\nabla_{\frac{\partial}{\partial \tau}}$ -covariant constant frame along $\pi_k^{-1}(c(s))$. The vector $W^i := (w_1^i, \dots, w_n^i)^T$, satisfies the equation

$$\left(\frac{\partial}{\partial \tau} + A(\tau, s) - A(\tau, 0)\right)W^i(\tau, s) = 0. \quad (21)$$

Integrating, we see that

$$W^i(4\pi, s) - W^i(0, s) = - \int_0^{4\pi} (A(u, s) - A(u, 0)) W^i(u, s) du. \quad (22)$$

Letting $H(s)$ denote the holonomy of the connection $\frac{\partial}{\partial \tau} + A(\tau, s)$, we have

$$\begin{aligned} H(s)w^i(0) &= w^i(4\pi, s) = w_j^i(4\pi, s)v_s^j(4\pi) = w_j^i(4\pi, s)H(0)_k^j v_s^k(0) \\ &= H(0)w^i + (w_j^i(4\pi, s) - w_j^i(0))H(0)_k^j v_s^k(0). \end{aligned} \quad (23)$$

Hence

$$\begin{aligned} |(H(s) - H(0))w^i(0)| &\leq |(W^i(4\pi, s) - W^i(0, s))| \\ &\leq \int_0^{4\pi} |A(u, s) - A(u, 0)| du \\ &\leq 4\pi|s| \sup_{(u,t) \in [0, 4\pi] \times [0, s]} |F(u, c(t))| \end{aligned} \quad (24)$$

Since the $\{w^i\}$ is an arbitrary unitary frame,

$$|(H(s) - H(0))| \leq 4\pi|s| \sup_{(u,t) \in [0, 4\pi] \times [0, s]} |F(u, c(t))| \quad (25)$$

In particular, the eigenvalues of the holonomy H around the S^1 fiber vary at most $O(|F|)$ in this region, and passing to the difference quotient, the covariant derivative of the holonomy is also $O(|F|)$.

Definition 7. Let $(E, A) \rightarrow \text{TN}_k$ denote a rank n Hermitian bundle with connection. Let $V \subset \mathbb{R}^3 \setminus \{\nu_1, \dots, \nu_k\}$ be any set. We say that A has κ -generic holonomy in V if on each circle fiber of $\pi_k^{-1}(V)$, the eigenvalues $\{l_j\}_j$ (defined mod \mathbb{Z}) of $\frac{1}{2\pi i} \ln(H)$ satisfy $\inf \{|l_j - l_m - k| : k \in \mathbb{Z}\} > \kappa$ for $j \neq m$. Equivalently

$$|(I - \text{Ad}(H))^{-1}|_{\text{sup}} < \frac{1}{\sqrt{2 - 2\cos(2\pi\kappa)}}, \quad (26)$$

where Ad denotes the adjoint representation, and $|\cdot|_{\text{sup}}$ denotes the sup norm.

Definition 8. We say that A has *generic asymptotic holonomy* if there exists a (unit speed parameterized) ray $\rho : [0, \infty) \rightarrow \mathbb{R}^3$ and there exist $t, \kappa > 0$ so that A has κ -generic holonomy in $\pi_k^{-1}(\rho([t, \infty)))$.

Suppose that A has κ generic holonomy on $\pi_k^{-1}(0)$ for a unit speed curve c in $\mathbb{R}^3 \setminus \{\nu_1, \dots, \nu_k\}$. Since

$$\begin{aligned} &|(I - \text{Ad}(H(s)))^{-1}|_{\text{sup}} - |(I - \text{Ad}(H(0)))^{-1}|_{\text{sup}} \\ &\leq |(I - \text{Ad}(H(s)))^{-1}|_{\text{sup}} |\text{Ad}(H(s) - H(0))|_{\text{sup}} |(I - \text{Ad}(H(0)))^{-1}|_{\text{sup}}, \end{aligned} \quad (27)$$

from (25) and manipulation, we see that

$$|(I - Ad(H(s)))^{-1}|_{sup} < \left(\sqrt{2 - 2\cos(2\pi\kappa)} - \sup_{\substack{u \in [0, 4\pi] \\ t \in [0, s]}} 4\pi|s| |ad(F)(u, c(t))|_{sup} \right)^{-1}. \quad (28)$$

Hence A has $\kappa_1 < \kappa$ generic holonomy on $\pi_k^{-1}(s)$ if

$$\sup_{(u, t) \in [0, 4\pi] \times [0, s]} 4\pi|s| |ad(F)(u, c(t))|_{sup} \leq \sqrt{2 - 2\cos(2\pi\kappa)} - \sqrt{2 - 2\cos(2\pi\kappa_2)}, \quad (29)$$

which is implied by the simpler (but not sharp) condition

$$\sup_{(u, t) \in [0, 4\pi] \times [0, s]} |s| |ad(F)(u, c(t))|_{sup} \leq \frac{1}{4}(\kappa - \kappa_1). \quad (30)$$

We see from (30) that if A has 2κ generic holonomy in $\pi_k^{-1}(\rho([t, \infty)))$, then for all $R < \infty$, there exists t_R such that A has κ generic holonomy in π_k^{-1} of the radius R tubular neighborhood of $\rho([t_R, \infty))$. Similarly, there exists a nondecreasing function, $R(t)$ so that for t large, A has $\frac{\kappa}{2}$ generic holonomy in π_k^{-1} of the radius $R(t)$ tubular neighborhood of $\rho([t, \infty))$. Suppose for simplicity that we choose the origin of \mathbb{R}^3 to be $\rho(0)$. Let

$$L(r) := \sup\{|ad(F(x))|_{sup} : |x| \geq r\}. \quad (31)$$

Then

$$R(t) \geq \frac{C\kappa}{L(r(\rho(t)))}, \quad (32)$$

where we can choose $C = \frac{1}{8}$ for κ sufficiently small. For simplicity, we henceforth choose

$$R(t) = \frac{\kappa}{8L(r(\rho(t)))}, \quad (33)$$

and κ sufficiently small.

3.2 First Bound

In this subsection, we exploit more features of the geometry of TN_k .

Lemma 9 (Minerbe [Min09]). *Let M^n , $n \geq 3$ be a complete connected Riemannian manifold, with nonnegative Ricci curvature. Assume there exists $o \in M$, $\nu > 2$, and $C_o > 0$ such that for all $t \geq s > 0$*

$$\frac{\text{Vol}(B(o, t))}{\text{Vol}(B(o, s))} \geq C_o \left(\frac{t}{s} \right)^\nu.$$

Then there exists $c_M > 0$ such that for all $f \in C_0^\infty(M)$,

$$c_M \left(\int_M |f|^{\frac{2n}{n-2}} d(x, o)^{-\frac{2}{n-2}} dv \right)^{1-\frac{2}{n}} \leq \|df\|_{L^2}^2. \quad (34)$$

Observe that TN_k satisfies the hypotheses of this lemma with $\nu = 3$ and $n = 4$. As r and the distance function are comparable at large distances, (34) yields

$$c_{\text{TN}_k} \left(\int_{\text{TN}_k} |f|^4 r^{-1} dv \right)^{\frac{1}{2}} \leq \|df\|_{L^2}^2. \quad (35)$$

Proposition 10. *Let $p \in \mathbb{R}^3 \setminus \{\nu_1, \dots, \nu_k\}$, with $|p| > 2R$. Let $U_R := \pi_k^{-1}(B(p, R))$. Assume that A is a Yang-Mills connection with κ -generic holonomy in U_R . Then there exists $C_M, C_\alpha > 0$ independent of p and R such that*

$$|F(x)| \leq C_M R^{-3/2} \text{ for } x \in U_{R/2} \setminus U_{R/4}, \quad (36)$$

and

$$|F(x)| \leq C_\alpha R^{\frac{4\alpha-1}{2}} \text{ for } x \in U_{R-R^\alpha} \setminus U_{R/4}, \forall \alpha \in (0, 1). \quad (37)$$

Proof. Let μ be a smooth monotone function satisfying $\mu(s) = 1$ for $s \leq 1$, $\mu(s) = 0$ for $s > 2$, and $|\mu'| \leq 2$. Let $r_p(x) := \max\{1, |p - \pi_k(x)|\}$. Let $\mu_j(t) := \mu(2^j \frac{(t - (R - R^\alpha))}{R^\alpha})$, for $\alpha \in (0, 1)$ to be determined. Then $\mu_j(r_p) = 1$ in $U_{R-R^\alpha+2^{-j}R^\alpha}$ and is supported in $U_{R-R^\alpha+2^{1-j}R^\alpha}$, with

$$|d\mu_j| \leq \frac{2^{1+j}}{R^\alpha} \chi_j, \quad (38)$$

where χ_j now denotes the characteristic function of $U_{R-R^\alpha+2^{1-j}R^\alpha}$. Along each circle fiber, we decompose $\text{ad}(E)$ into $Z \oplus B$, where Z is the centralizer of Φ and B is the orthogonal complement of Z . In particular, B is the span of the nonzero eigenspaces of $\text{ad}(\Phi)$ (as in Definition 7). Observe that Z is abelian by the κ -generic hypothesis. With respect to this decomposition, decompose the curvature as

$$F_A = F_A^Z + F_A^B. \quad (39)$$

Choose $\eta = r_p^\alpha \mu_1(r_p)$. With this choice, (6) becomes

$$\begin{aligned} \|d(r_p^\alpha \mu_1)F\|^2 &= \|\nabla(\mu_1 r_p^\alpha F)\|^2 - \langle R_{ij}(r_p^\alpha \mu_1)^2 F, F_{ik} e^j \wedge e^k \rangle \\ &\quad - 3 \langle [F_{ij}^Z, (r_p^\alpha \mu_1)^2 F_{jk}^B], F_{ik}^B \rangle - \langle [F_{ij}^B, (r_p^\alpha \mu_1)^2 F_{jk}^B], F_{ik}^B \rangle. \end{aligned} \quad (40)$$

The generic holonomy assumption and the Fourier expansion yield

$$\int_{S^1} |\nabla_{\sqrt{V} \frac{\partial}{\partial \tau}} \mu_1 r_p^\alpha F|^2 d\tau \geq \beta^2 \int_{S^1} |\kappa \mu_1 r_p^\alpha F^B|^2 d\tau, \quad (41)$$

with $\beta^2 := \inf \frac{V}{4}$. On TN_k there exists $c > 0$ with $|R_{ij} r^3| < c$. This implies

$$\|\nabla(\mu_1 r_p^\alpha F)\|^2 \leq c_1 \|F_A\|^2, \quad (42)$$

with c_1 independent of R and p , for R large.

Using (34) and (42) we deduce that

$$\int |\mu_1 F_A|^4 r_p^{4\alpha-1} dv < c_2 \|F_A\|^2, \quad (43)$$

for c_2 independent of p and R , for R large. Now set up the induction procedure.

Take $\eta = \mu_j r_p^{k_j} |F_A|^{p_j}$ in (6) and reorganize to get

$$\begin{aligned} \|\nabla(\mu_j r_p^{k_j} |F_A|^{p_j} F_A)\|^2 &\leq \|d(\mu_j r_p^{k_j} |F_A|^{p_j})|F_A\|^2 + c\|r^{-3/2}\eta F_A\|^2 \\ &+ \langle [F_{ij}, \eta F_{jk}], \eta F_{ik} \rangle = \|d(\frac{\mu_j r_p^{k_j} p_j}{p_j + 1} |F_A|^{p_j+1})\|^2 + \|\frac{|F_A|^{p_j+1}}{p_j + 1} d(\mu_j r_p^{k_j})\|^2 \\ &+ \frac{2p_j}{(p_j + 1)^2} \langle d(\mu_j r_p^{k_j} |F_A|^{p_j+1}), |F_A|^{p_j+1} d(\mu_j r_p^{k_j}) \rangle \\ &+ c\|r^{-3/2}\eta F_A\|^2 + \langle [F_{ij}, \eta F_{jk}], \eta F_{ik} \rangle. \end{aligned} \quad (44)$$

Sharpening Kato's inequality and exploiting (41), we have, again integrating over the fibers,

$$\begin{aligned} &\int_{S^1} |\nabla(\mu_j r_p^{k_j} |F_A|^{p_j} F_A)|^2 d\tau \\ &= \int_{S^1} |d(\mu_j r_p^{k_j} |F_A|^{p_j+1})|^2 d\tau + \int_{S^1} |\mu_j r_p^{k_j} |F_A|^{p_j+1} \nabla(\frac{F_A}{|F_A|})|^2 d\tau \\ &\geq \int_{S^1} |d(\mu_j r_p^{k_j} |F_A|^{p_j+1})|^2 d\tau + \beta^2 \kappa^2 \int_{S^1} |\mu_j r_p^{k_j} |F_A|^{p_j} F_A^B|^2 d\tau. \end{aligned} \quad (45)$$

Combining (45) with (44), applying Cauchy-Schwartz to the cross terms, and absorbing the commutator terms into the κ terms for R large gives

$$\begin{aligned} &\frac{1}{p_j + 1} \|d(\mu_j r_p^{k_j} |F_A|^{p_j+1})\|^2 + \frac{\beta^2 \kappa^2}{2} \|\mu_j r_p^{k_j} |F_A|^{p_j} F_A^B\|^2 \\ &\leq \frac{2p_j + 1}{(p_j + 1)^2} \| |F_A|^{p_j+1} d(\mu_j r_p^{k_j}) \|^2 + c\|\mu_j r_p^{k_j} r^{-3/2} |F_A|^{p_j+1}\|^2. \end{aligned} \quad (46)$$

Applying (34) and $|d(\mu_j r_p^{k_j})| \leq (2^{1+j} + k_j) \chi_j r_p^{k_j - \alpha}$, we obtain,

$$\|\chi_{j+1} r_p^{\frac{4k_j-1}{4p_j+4}} F_A\|_{L^{4p_j+4}}^{2p_j+2} \leq c_M [2(2^{1+j} + k_j)^2 + c(p_j + 1)] \|\chi_j r_p^{\frac{k_j-\alpha}{p_j+1}} F_A\|_{L^{2p_j+2}}^{2p_j+2}. \quad (47)$$

Now set $b_j := 2p_j + 2$, $b_{j+1} := 2b_j$, $a_j = \frac{2k_j-2\alpha}{b_j}$, $a_{j+1} = a_j + \frac{4\alpha-1}{2b_j}$, and induct with (47) to obtain

$$\|\chi_{m+1} r_p^{a_m + \frac{4\alpha-1}{2b_m}} F_A\|_{L^{b_{m+1}}} \leq \left(\prod_{1 \leq j \leq m} (c_M [2(2^{1+j} + a_j b_j)^2 + c b_j])^{\frac{1}{b_j}} \right) \|\chi_1 r_p^{a_1} F_A\|_{L^{b_1}}. \quad (48)$$

We start the recurrence relations with $b_1 = 4$ and $a_1 = \frac{3}{4}$. Taking the limit as $m \rightarrow \infty$ and using (43) gives

$$\|\chi_{U_{R-R\alpha}} r_p^{\frac{4\alpha-1}{2}} F_A\|_{L^\infty} \leq \left(\prod_{1 \leq j \leq \infty} (c_M [2(2^{1+j} + a_j b_j)^2 + c b_j])^{\frac{1}{b_j}} \right) c_2 \|F_A\|_{L^2}^2. \quad (49)$$

Since the infinite product on the right hand side is convergent, we obtain a pointwise bound for $\left| r_p^{\frac{4\alpha-1}{2}} F_A \right|$ independent of p and R for R large. If we replace $\frac{t-(R-R^\alpha)}{R^\alpha}$ in the definition of μ_j by $\frac{t-\frac{R}{2}}{R}$, the preceding computation gives $\left| r_p^{\frac{3}{2}} F_A \right|$ is bounded on $U_{\frac{R}{2}}$. The desired pointwise bounds are valid on annuli on which r_p is comparable to R . \square

Proposition 11. *Assume that $F_A \in L^2$ is a Yang-Mills curvature and that A has generic asymptotic holonomy. Then $r^{3/2}|F_A|$ is bounded.*

Proof. Let A have 4κ generic holonomy along a ray ρ . Choose the origin of \mathbb{R}^3 to be $\rho(0)$. Let $U(s, t)$ denote π_k^{-1} of the radius s tubular neighborhood of $\rho([t, \infty))$. Fix some R_1 large. Guided by (30), we fix t_1 large so that

$$L(r(\rho(t_1))) \leq \frac{\kappa}{4R_1}. \quad (50)$$

The preceding proposition applied to each $p = \rho(t)$, for $t \geq t_{R_1} + 2R_1$, implies that

$$|F|(x) \leq c_M R_1^{-3/2} \text{ in } U\left(\frac{R_1}{2}, t_{R_1} + 3R_1\right) \quad (51)$$

$$|F|(x) \leq c_\alpha R_1^{\frac{-4\alpha+1}{2}} \text{ in } U(R_1 - R_1^\alpha, t_{R_1} + 3R_1). \quad (52)$$

These bounds and (30) imply that A has $4\kappa - 2c_M R_1^{-1/2}$ generic holonomy in $U(\frac{R_1}{2}, t_{R_1} + 3R_1)$, and $4\kappa - 4c_\alpha R_1^{\frac{3}{2}-2\alpha}$ generic holonomy in $U(R_1 - R_1^\alpha, t_{R_1} + 3R_1)$. Hence each ray $\rho_v(t) := \rho(t) + v$, $v \perp \rho'$, $|v| < \frac{R_1}{2}$ in $\pi_k(U(\frac{R_1}{2}, t_{R_1} + 3R_1))$, parallel to ρ , has $4\kappa - 2c_M R_1^{-1/2}$ generic holonomy. By (33) we have $2\kappa - c_M R_1^{-1/2}$ generic holonomy in $v + U(\frac{2\kappa - c_M R_1^{-1/2}}{2\kappa} R_1, t_{R_1} + 2R_1)$. Taking the union of all these tubular neighborhoods yields $2\kappa - c_M R_1^{-1/2}$ generic holonomy on $U(\frac{3}{2}R_1 - \frac{c_M}{2\kappa} R_1^{1/2}, t_{R_1} + 2R_1)$. Similarly, A has $2\kappa - 2c_\alpha R_1^{\frac{3}{2}-2\alpha}$ generic holonomy in $U(2R_1 - R_1^\alpha - \frac{c_\alpha R_1^{\frac{5}{2}-2\alpha}}{\kappa}, t_{R_1} + 2R_1)$. This suggests choosing $\alpha = \frac{5}{6}$ so the latter cylinder becomes $U(2R_1 - (1 + \frac{c_\alpha}{\kappa}) R_1^{\frac{5}{6}}, t_{R_1} + 2R_1)$, with $2\kappa - 2c_{\frac{5}{6}} R_1^{-\frac{1}{6}}$ generic holonomy.

Preparing the induction we set

$$R_{j+1} := R_1 + R_j - R_j^{\frac{5}{6}} \left(1 + \frac{2c_{\frac{5}{6}}}{2\kappa_j}\right), \quad (53)$$

and

$$\kappa_{j+1} := \kappa - c_{\frac{5}{6}} R_j^{-\frac{1}{6}}. \quad (54)$$

The preceding argument implies that for R_1 sufficiently large, A has κ generic

holonomy in $U(R_j, t_{R_1} + 3R_1)$ for $j = 2$. Then Proposition 10 implies

$$|F|(x) \leq c_M R_j^{-3/2} \text{ in } U\left(\frac{R_j}{2}, t_{R_1} + 3R_1\right) \quad (55)$$

$$|F|(x) \leq c_\alpha R_j^{\frac{-4\alpha+1}{2}} \text{ in } U(R_j - R_j^\alpha, t_{R_1} + 3R_1). \quad (56)$$

Now we perform the induction on j . Repeating the R_1 estimates for R_j (but noting that our bound for $L(r(\rho(t_1)))$ depends on R_1 and not R_j - hence the definition of R_{j+1}), we see that A has $2\kappa_j - c_M R_j^{-1/2}$ generic holonomy in $U(R_1 + \frac{R_j}{2} - \frac{c_M}{2\kappa} R_j^{1/2}, t_{R_1} + 2R_1)$ and κ_{j+1} generic holonomy in $U(R_{j+1}, t_{R_1} + 2R_1)$. The induction terminates when $r(\rho(t_1)) \leq 2R_j$ and (55) implies $r^{3/2}|F_A| \leq 8c_M$ when $r(\rho(t_1)) \leq 2R_j$. We extend the induction by increasing t_1 . This yields the asserted bound in a cone of fixed cone angle around ρ . We now repeat the process for every ray in the boundary of the cone. We continue in this fashion until we deduce κ generic holonomy in the complement of a compact set and the consequent curvature decay. \square

3.3 Quadratic Curvature Decay

In this subsection we sharpen our curvature decay to quadratic decay. We first recall the Kato-Yau inequality for anti-self-dual forms and a Hardy inequality for TN_k .

Lemma 12 (Kato-Yau [IN90, CGH00]). *Let h be a closed anti-self-dual form, over a four-manifold, with coefficients in a Hermitian bundle, then for any unit vector u*

$$\frac{3}{2}|\nabla_u h|^2 \leq |\nabla h|^2. \quad (57)$$

Proof. Let $\{e_1, e_2, e_3, e_4\}$ be an oriented orthonormal frame. The Bianchi identities combined with anti-self-duality imply

$$\begin{aligned} -h_{12;1} + h_{13;4} + h_{41;3} &= 0, & h_{12;2} + h_{13;3} + h_{14;4} &= 0, \\ h_{12;3} + h_{31;2} - h_{14;1} &= 0, & h_{12;4} - h_{31;1} + h_{41;2} &= 0. \end{aligned}$$

Hence

$$|h_{12;1}|^2 + |h_{13;1}|^2 + |h_{14;1}|^2 \leq 2(|h_{13;4}|^2 + |h_{41;3}|^2 + |h_{12;4}|^2 + |h_{41;2}|^2 + |h_{12;3}|^2 + |h_{31;2}|^2).$$

Hence

$$\frac{3}{2}|\nabla_1 h|^2 \leq |\nabla h|^2.$$

Choose $e_1 = u$, and the lemma follows. \square

Lemma 13 (Hardy inequality for TN_k). *Let $f \in C_0^\infty(\text{TN}_k)$ then*

$$\|df\|^2 \geq \frac{1}{4}\|r^{-1}f\|^2 - C_H\|(r+1)^{-3/2}f\|^2. \quad (58)$$

If f is compactly supported outside a compact set K containing the $\{\nu_j\}_{j=1}^k$, then we have

$$\left\| \frac{\partial f}{\partial r} \right\|^2 \geq \frac{1}{4} \|r^{-1}f\|^2 - C_H \|(r+1)^{-3/2}f\|^2. \quad (59)$$

Proof. The proof is a minor modification of the proof of Proposition 3.7 in [DS13] and extends to any ALF space. Let r again denote the \mathbb{R}^3 radial distance and denote by $d\Omega$ the Euclidean volume form on a unit sphere S^3 . Outside a compact set K containing the $\{\nu_j\}_j$, the volume form of TN_k satisfies $d\text{Vol} = Vr^2 dr \wedge d\Omega$ where $V = l + O(r^{-1})$ and $|dV| = O(r^{-2})$. Let $\eta(r)$ be a cutoff function vanishing on $B_T \supset \pi_k(K)$, identically one in $\mathbb{R}^3 \setminus B_{T+1}$, and satisfying $|d\eta| \leq 2$, $|\Delta\eta| \leq 8$.

$$\begin{aligned} \left\| \frac{\eta f}{r} \right\|^2 &= \int_T^\infty \int_{S^3} |\eta f|^2 V dr d\Omega \\ &= - \int_T^\infty \int_{S^3} 2r\eta f \frac{\partial \eta f}{\partial r} V dr d\Omega - \int_T^\infty \int_{S^3} r|\eta f|^2 \frac{\partial V}{\partial r} dr d\Omega \\ &\leq \frac{1}{2} \left\| \frac{\eta f}{r} \right\|^2 + 2 \left\| \frac{\partial \eta f}{\partial r} \right\|^2 + C \|r^{-3/2}\eta f\|^2. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{4} \left\| \frac{\eta f}{r} \right\|^2 &\leq \left\| \frac{\partial \eta f}{\partial r} \right\|^2 + \frac{C}{2} \|r^{-3/2}\eta f\|^2 \\ &\leq \left\| \eta \frac{\partial f}{\partial r} \right\|^2 + \|f d\eta\|^2 + \frac{1}{2} \int f^2 \Delta \eta^2 dv + \frac{C}{2} \|r^{-3/2}\eta f\|^2, \end{aligned}$$

and the results follow. \square

Theorem 14. *Let A be an instanton with generic asymptotic holonomy. Then $\|r^2 F_A\|_{L^\infty} < \infty$.*

Proof. We first prove $r^a |F_A| \in L^2$, for any $a < \frac{1}{2}$. We begin again with (6):

$$0 = \|\nabla(\eta F_A)\|^2 - \|d\eta|F_A\|^2 - \langle \varepsilon^i \varepsilon_j^* R_{ij} \eta F, \eta F \rangle - \langle [F_{ij}, \eta F_{jk}], \eta F_{ik} \rangle. \quad (60)$$

This equation holds for all η with $\eta F \in H^1$. Set $r_n = r$, for $r \leq n$ and $r_n = n$, for $r \geq n$. Choosing $\eta = \eta_n(r) = r_n^p r^{\frac{1}{2}}$ with $p \leq 1$ yields

$$\|\nabla(r_n^p r^{\frac{1}{2}} F_A)\|^2 = \left\| d(r_n^p r^{\frac{1}{2}}) |F_A| \right\|^2 + \langle \varepsilon^i \varepsilon_j^* R_{ij} r_n^{2p} r F, F \rangle + \langle [F_{ij}, r_n^{2p} r F_{jk}], F_{ik} \rangle. \quad (61)$$

Write

$$\|\nabla(\eta F_A)\|^2 = \|\nabla^0(\eta F_A)\|^2 + \|\nabla_{\hat{e}_1}(\eta F_A)\|^2,$$

where \hat{e}_1 is a unit vector in the radial direction and ∇^0 denotes the summand of the covariant derivative in the directions orthogonal to \hat{e}_1 . The Hardy inequality (59) and the Kato inequality give

$$\|\nabla_{\hat{e}_1}(\eta F_A)\|^2 \geq \frac{1}{4} \left\| \frac{\eta F_A}{r} \right\|^2 - C_H \left\| \frac{\eta F_A}{(r+1)^{3/2}} \right\|^2. \quad (62)$$

In the proof of Proposition 11 we proved $rF^B \in L^2$. Hence

$$\langle [F_{ij}, \eta_n F_{jk}], \eta_n F_{ik} \rangle(x) \leq C_1 |F_A| |rF^B|^2 \frac{\eta_n^2}{r^2} \leq C_3 |rF^B|^2 \frac{\eta_n^2}{r^{7/2}} \in L^1,$$

with L^1 bound independent of n since $p \leq 1$. Combining this estimate and (62) with (61) gives

$$\frac{1}{4} \|r_n^p r^{-\frac{1}{2}} F_A\|^2 + \|r_n^p r^{\frac{1}{2}} \nabla^0 F_A\|^2 \leq \|d(r_n^p r^{\frac{1}{2}}) F_A\|^2 + C_2 \|r_n^p r^{-1} F_A\|^2 + C_4. \quad (63)$$

Here we have used the cubic decay of R_{ij} .

Now we use the Bianchi identity to estimate $\|r_n^p r^{\frac{1}{2}} \nabla^0 F_A\|^2$ from below. Let $\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4$ be an oriented orthonormal frame around a point $x \in \text{TN}_k$, with \hat{e}_1 the unit radial vector field. The Bianchi identity implies

$$0 = (F_{1i;j} - F_{1j;i} - F_{1m;1}, F_{1m}) \quad (64)$$

Here $\{i, j, m\} = \{2, 3, 4\}$ and (\cdot, \cdot) denotes the fiber inner-product. Integrating this equality and using

$$(F_{1m;1}, F_{1m}) r_n^{2p} = \frac{1}{2} \text{div} (r_n^{2p} |F_{1m}|^2 \hat{e}_1) - |F_{1m}|^2 (p\chi_n + 1) \frac{r_n^{2p}}{r} + O(|F_{1m}|^2 r_n^{2p} r^{-2})$$

and applying Cauchy-Schwartz yields

$$\sqrt{2} \|r_n^p r^{1/2} \nabla^0 F_A\| \|r_n^p r^{-1/2} F_A\| \geq \int |F_A|^2 (pt_{n,p} + 1) r_n^{2p} r^{-1} dv - c, \quad (65)$$

where $t_{n,p} = \frac{\int |F_A|^2 \chi_n r_n^{2p} r^{-1} dv}{\int |F_A|^2 r_n^{2p} r^{-1} dv}$. Squaring both sides yields

$$\|r_n^p r^{1/2} \nabla^0 F_A\|^2 \geq \frac{(pt_{n,p} + 1)^2}{2} \int |F_A|^2 r_n^{2p} r^{-1} dv - c_2. \quad (66)$$

Inserting (66) into (63) gives

$$\frac{1}{2} [1 - p^2 + p^2 (t_{n,p} - 1)^2] \int |F_A|^2 r_n^{2p} r^{-1} dv \leq c_3. \quad (67)$$

Hence, for all $p < 1$, we may take the limit as $n \rightarrow \infty$ in (67) to deduce

$$\int |F_A|^2 r^{2p-1} dv < \infty.$$

Taking $p = 1$, we observe that if $\limsup_{n \rightarrow \infty} (t_{n,1} - 1)^2 \neq 0$ then $\|r^{\frac{1}{2}} F_A\|_{L^2}^2 < \infty$. Additionally, we have the useful consequence of (67), for $p < 1$

$$\int |F_A|^2 r^{2p-1} dv \leq \frac{c_3}{1-p}. \quad (68)$$

Setting $p = 1 - \frac{1}{\ln(n)}$ in Equation (68) gives

$$\int_{r \leq n^k} |F_A|^2 r dv \leq c_3 e^{2k} \ln(n). \quad (69)$$

Given any $T > 2$, and setting $T^N = n$, and $k = 2$, (69) becomes

$$\frac{1}{N} \sum_{m=1}^{2N-1} \int_{T^m \leq r \leq T^{m+1}} |F_A|^2 r dv \leq c_3 e^4 \ln(T). \quad (70)$$

Thus, we see that the average value of $\int_{T^m \leq r \leq T^{m+1}} |F_A|^2 r dv$ is less than $c_3 e^4 \ln(T)$. For each m such that we have $\int_{T^m \leq r \leq T^{m+1}} |F_A|^2 r dv < c_3 e^4 \ln(T)$, local Moser iteration implies that

$$r^2 |F_A|(x) \leq C(T) \quad (71)$$

on the spherical annulus $T^m + T^{m-1} \leq r(x) \leq T^{m+1} - T^m$. Because (70) holds for all large N , there is a uniform pointwise upper bound for $r^2 |F_A|$ on infinitely many spherical annuli. If $r^{1+p} |F_A|$ is not uniformly bounded for all $p \leq 1$, then it must achieve local maxima between some of these annuli, for some p . At such a maximum point we have

$$0 = (1+p)|F_A| + r|F_A|_r, \quad (72)$$

and

$$\begin{aligned} 0 \geq & -\frac{1}{2} \Delta(|F_A|^2 r^{2+2p}) = r^{2+2p} |\nabla F_A|^2 + \frac{(2p^2 + 5p + 3)}{V} r^{2p} |F_A|^2 \\ & + \frac{(2+2p)r^{2p+1}}{V} (|F_A|^2)_r + O(r^{2p-1} |F_A|^2) - r^{2+2p} \langle [F_{ij}, F_{jk}], F_{ik} \rangle. \end{aligned} \quad (73)$$

Substituting (72) in (73), dividing by r^{2p} , noticing that $V^{-1} < l^{-1}$ and using the anti-self dual Kato-Yau inequality (57) and the preceding lemma, we find for any $\epsilon > 0$,

$$\frac{1-p^2}{2} |F_A|^2 \leq O(r^{-1} |F_A|^2) + O(r^{-5+\epsilon}). \quad (74)$$

Hence, where a local maximum occurs we must have $1-p = O(r^{\epsilon-1})$ or $|F_A| < r^{-2}$.

For $p = 1 - \frac{1}{N}$ and N large, $(1-p) = O(\frac{1}{\ln(r)})$ when $T^N \leq r \leq T^{4N}$. Therefore $r^{2+2p} |F_A|^2$ has no local maximum for $T^N \leq r \leq T^{4N}$, unless $|F_A| < r^{-2}$ at the maximum. In this range, $r^{2+2p} = r^4 e^{-\ln(r)(1-p)} \in [r^4 T^{-4}, r^4 T^{-1}]$.

Suppose now that $r^2 |F_A|^2$ is unbounded and $T < 4$. Then there exists a sequence of integers m with $m \rightarrow \infty$ such that $\sup_{T^m \leq r(x) \leq T^{m+1}} r^{2+2p} |F_A|^2(x) > 10^6 C(T)$, with $C(T)$ the constant given by Moser iteration in (71), and $p = 1 - \frac{1}{m}$. Since $r^{2+2p} |F_A|^2(x)$ has no local maximum in $T^m \leq r(x) \leq T^{4m}$, we must have for every $R \in [T^m, T^{4m}]$, there exists an x such that $r(x) = R$ and $r^4(x) |F_A|^2(x) > 10^5 C(T) T^{-4}$. Hence, by (71) there exists no integer

$j \in [0, 3m - 1]$ such that $\int_{T^{m+j} \leq r \leq T^{m+1+j}} |F_A|^2 r dv < c_3 e^2 \ln(T)$. This contradicts (70). \square

Corollary 15. *Let A be an instanton with generic asymptotic holonomy. Then $\|r^3 \nabla F_A\|_{L^\infty} < \infty$.*

Proof. Given x large, let $R = \frac{|x|}{2}$, and let η be a cutoff function supported in $\pi_k^{-1}(B_R(x))$ and identically one in $\pi_k^{-1}(B_{\frac{R}{2}}(x))$, with $0 \leq \eta \leq 1$ and $|d\eta| \leq \frac{8}{R}$. Then the Bochner formula (6) applied to ηF_A implies

$$\int_{\pi_k^{-1}(B_{\frac{R}{2}}(x))} |\nabla F_A|^2 dv \leq \frac{c}{R^3}.$$

Hence $\|r^{3/2} \nabla F_A\|^2$ is uniformly bounded on such balls. The proof now follows from Moser iteration as in the proof of Proposition 10. We replace the cutoff function μ in that proof with a function with $|d\mu| = O(r^{-1})$. This allows us to set the parameter α in that proof to 1. Setting $a_1 = \frac{3}{2}$ and $b_1 = 2$ gives $|r^3 \nabla F_A|$ is bounded. \square

4 Asymptotic Form of an Instanton

Let Σ be a smooth local section of the S^1 bundle over $\mathbb{R}^3 \setminus \bigcup_{\sigma=1}^k \rho_\sigma$, with $\{\rho_\sigma\}_{\sigma=1}^k$ nonintersecting rays with each ρ_σ starting at ν_σ . Given $\vec{x} \in \mathbb{R}^3 \setminus \bigcup_{\sigma=1}^k \rho_\sigma$, let $S_{\vec{x}}^1 := \pi_k^{-1}(\vec{x})$ denote the fiber over \vec{x} and let $\{e^{4\pi i \mu_j(\vec{x})}\}_j$ be the eigenvalues of the holonomy along the circle fiber $S_{\vec{x}}^1$, with base point $\Sigma(\vec{x}) \in S_{\vec{x}}^1$. Let $\{y_j\}_j$ form a unitary eigenframe on Σ , and let $\{y_j(\tau)\}_j$ denote their parallel translates in the fiber. Then $\{e^{-i\tau \mu_j(\vec{x})} y_j(\tau) =: w_j(\tau)\}$ gives a continuous frame on $\pi_k^{-1}(\Sigma)$. Writing $y_j(\tau) = e^{i\tau \mu_j(\vec{x})} w_j(\tau)$, we see, as usual, that the $\mu_j + \frac{1}{2}\mathbb{Z}$ are independent of the base point $\Sigma(\vec{x})$, on any given fiber. The frame $\{w_j\}$ is particularly convenient, as in that frame $A\left(\frac{\partial}{\partial \tau}\right)$ is diagonal, moreover,

$$A\left(\frac{\partial}{\partial \tau}\right) = -i \operatorname{diag}(\mu_j).$$

We now consider the variation of the μ_j as a function of \vec{x} . Observe that $(\nabla_{\partial_k} w_j, w_j) + (w_j, \nabla_{\partial_k} w_j) = 0$ implies that $(\nabla_{\partial_k} w_j, \nabla_{\partial_\tau} w_j) = -i \mu_j (\nabla_{\partial_k} w_j, w_j)$

is real.

$$\begin{aligned}
\partial_k \mu_j(\vec{x}) &= -\partial_k \frac{1}{2\pi} \text{Im} \int_{S_{\vec{x}}^1} (\nabla_{\partial_\tau} w_j, w_j) d\tau = -\text{Im} \frac{1}{2\pi} \int_{S_{\vec{x}}^1} (\nabla_{\partial_k} \nabla_{\partial_\tau} w_j, w_j) d\tau \\
&= -\frac{1}{2\pi} \int_{S_{\vec{x}}^1} (F(\partial_k, \partial_\tau) w_j, w_j) d\tau - \frac{1}{2\pi} \text{Im} \int_{S_{\vec{x}}^1} (\nabla_{\partial_\tau} \nabla_{\partial_k} w_j, w_j) d\tau \\
&= -\frac{1}{2\pi} \int_{S_{\vec{x}}^1} (F(\partial_k, \partial_\tau) w_j, w_j) d\tau + \frac{1}{2\pi} \text{Im} \int_{S_{\vec{x}}^1} (\nabla_{\partial_k} w_j, \nabla_{\partial_\tau} w_j) d\tau \\
&= -\frac{1}{2\pi} \int_{S_{\vec{x}}^1} (F(\partial_k, \partial_\tau) w_j, w_j) d\tau.
\end{aligned}$$

Hence

$$|d\mu_j| < \frac{c}{r^2}. \quad (75)$$

Let $\bar{\mu}_j(r)$ denote the average of μ_j over the sphere of radius r in \mathbb{R}^3 . Then (75) implies

$$\mu_j = \bar{\mu}_j(\infty) + \delta_j, \quad (76)$$

with $\delta_j = O(\frac{1}{r})$.

Definition 16. We denote by F_A^0 the zero Fourier mode of F_A^Z in the $\{w_j\}_j$ frame.

Proposition 17. *Let A be an instanton with generic asymptotic holonomy. Then $\|r^3 F_A^B\|_{L^\infty} < \infty$ and $\|r^3(F_A - F_A^0)\|_{L^\infty} < \infty$.*

Proof. By the Corollary 15, $\sum_{k \in \mathbb{Z}} |(k + iad(\mu))F_k|^2 < Cr^{-6}$, where F_k denotes the k^{th} Fourier coefficient in the $\{w_j\}$ frame. Hence

$$|\sum_{k \in \mathbb{Z}} F_k^B e^{\frac{ik}{2}\tau}| \leq \sqrt{\sum_{k \in \mathbb{Z}} |(k + iad(\mu))F_k^B|^2} \sqrt{\sum_{k \in \mathbb{Z}} \inf_{i \neq j} (k + \mu_i - \mu_j)^{-2}} \leq C'r^{-3}.$$

The proof of the second claim is identical. \square

We now sharpen (76). Differentiating the relation $\nabla_{\partial_\tau} w_j = -i\mu_j w_j$ gives

$$(\nabla_{\partial_\tau} + i\mu_j) \nabla_{\partial_k} w_j = -i\partial_k(\mu_j)w_j - F(\partial_k, \partial_\tau)w_j.$$

Hence

$$\partial_\tau \langle \nabla_{\partial_k} w_j, w_m \rangle + i(\mu_j - \mu_m) \langle \nabla_{\partial_k} w_j, w_m \rangle = -i\partial_k(\mu_j)\delta_{jm} - \langle F(\partial_k, \partial_\tau)w_j, w_m \rangle.$$

Generic asymptotic holonomy and Proposition 17 imply that for $m \neq j$,

$$|\langle \nabla w_j, w_m \rangle| = O\left(\frac{1}{r^3}\right).$$

We have further freedom to choose a frame that is radially covariantly constant on Σ for the complex line bundle spanned by w_j so that $|\langle \nabla w_j, w_j \rangle|$ vanishes at a given point and is $O(\frac{1}{r^2})$ in a neighborhood of the point.

We can refine (76) by taking a second derivative. Choose coordinates so that the connection form ω satisfies

$$|\nabla^j \omega| = O(r^{-1-j}), \text{ for } j = 0, 1. \quad (77)$$

$$\begin{aligned} \sum_k \frac{\partial^2 \mu_j}{(\partial x^k)^2} &= \sum_k -\frac{1}{4\pi} \text{Im} \int_{S_x^1} ((\nabla_{\partial_k} F(\partial_k, \partial_\tau)) w_j, w_j) d\tau \\ &\quad - \frac{1}{4\pi} \text{Im} \int_{S_x^1} (F(\partial_k, \partial_\tau) \nabla_{\partial_k} w_j, w_j) d\tau - \frac{1}{4\pi} \text{Im} \int_{S_x^1} (F(\partial_k, \partial_\tau) w_j, \nabla_{\partial_k} w_j) d\tau \\ &= \sum_k \frac{1}{4\pi} \text{Im} \int_{S_x^1} (F(\nabla_{\partial_k} \partial_k, \partial_\tau) w_j, w_j) d\tau + \frac{1}{4\pi} \text{Im} \int_{S_x^1} (F(\partial_k, \nabla_{\partial_k} \partial_\tau) w_j, w_j) d\tau \\ &\quad + \frac{1}{2\pi} \text{Im} \int_{S_x^1} (\nabla_{\partial_k} w_j, F(\partial_k, \partial_\tau) w_j) d\tau = O(r^{-4}). \end{aligned}$$

For a self-dual connection, we can sharpen this estimate.

$$\begin{aligned} F(\nabla_k \partial_k, \partial_\tau) + F(\partial_k, \nabla_k \partial_\tau) &= \Gamma_{kk}^i F(p_i, \partial_\tau) + \Gamma_{k\tau}^i F(\partial_k, \partial_i) + \Gamma_{k\tau}^\tau F(\partial_k, \partial_\tau) \\ &= V^{-1} V_k F(p_k, \partial_\tau) - \frac{3}{2} V^{-1} V_k F(p_k, \partial_\tau) + V^{-1} (V^{-1} \omega_i)_k F(\partial_k, \partial_i) - \frac{1}{2} V^{-1} V_k F(\partial_k, \partial_\tau) \\ &= -V^{-1} V_j F(p_j, \partial_\tau) - V^{-3} V_k \omega_i F(\partial_k, \partial_i) + \frac{1}{2} V^{-2} (d\omega)_{ki} F(\partial_k, \partial_i) \\ &= -V^{-1} V_j F(p_j, \partial_\tau) - V^{-3} V_k \omega_i F(\partial_k, \partial_i) + \frac{1}{2} V^{-2} (*_3 dV)_{ki} F(\partial_k, \partial_i). \end{aligned}$$

Rotate coordinates so that ∇V is in the ∂_1 direction to get

$$-V^{-1} V_1 F(V^{-1/2} p_1, V^{1/2} \partial_\tau) - V^{-3} V_k \omega_i F(\partial_k, \partial_i) + V^{-1} V_1 F(V^{-1/2} \partial_2, V^{-1/2} \partial_3) = O(r^{-5}).$$

Therefore

$$\Delta \mu_j = O(r^{-4}), \quad (78)$$

for an anti-self-dual connection and

$$\Delta \mu_j = O(r^{-5}), \quad (79)$$

for a self dual connection. Moreover, μ_j descends to a function on the base \mathbb{R}^3 with $\Delta_{\mathbb{R}^3} \mu_j = O(r^{-4})$ or $O(r^{-4})$ in the anti self dual and self dual cases respectively. In this computation, if we considered instead $\frac{\partial^2 \mu_j}{\partial x^k \partial x^m}$, then there would be an $\nabla_{\frac{\partial}{\partial x^m}} (F(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial \tau}))$ term which we can not reduce using the Yang Mills equation. Hence, in this case, we have

$$\frac{\partial^2 \mu_j}{\partial x^k \partial x^m} = O(r^{-3}). \quad (80)$$

Theorem 18. *For a finite action self-dual connection with generic asymptotic holonomy there are real constants $\lambda_j = 2l\bar{\mu}_j(\infty)$ and ϑ_j , such that*

$$\mu_j = \frac{\lambda_j}{2l} + \frac{\vartheta_j}{r} + O\left(\frac{1}{r^2}\right). \quad (81)$$

For a finite action anti-self-dual connection with generic asymptotic holonomy there are real constants $\lambda_j = 2l\bar{\mu}_j(\infty)$ and m_j , such that

$$\mu_j = \frac{\lambda_j}{2l} + \frac{\vartheta_j}{r} + O\left(\frac{\ln(r)}{r^2}\right). \quad (82)$$

Moreover,

$$d(\mu_j - \frac{\vartheta_j}{r}) = O\left(\frac{\ln(r)}{r^3}\right). \quad (83)$$

In $\{w_j\}$ frame the anti-self-dual connection A has the form

$$A = -i \text{diag} \left(\frac{\lambda_j + \frac{m_j}{r}}{2V} (d\tau + \omega) - \frac{m_j}{k} \omega \right) + O(\ln r / r^2), \quad (84)$$

with $m_j = l\vartheta_j + \lambda_j k$ integer.

Proof. Let $\lambda_j := 2l\bar{\mu}_j(\infty)$, so that the asymptotic values of the holonomy are $\exp(2\pi i \lambda_j / l)$, and let $\hat{\mu}_j := \eta \times (\mu_j - \bar{\mu}_j(\infty))$, where η is a smooth function supported in a neighborhood of ∞ and identically one in a smaller neighborhood of ∞ .

Using the Newtonian potential on \mathbb{R}^3 , one has $\hat{\mu}_j(x) = \frac{1}{4\pi} \int \frac{1}{|x-y|} \Delta_{\mathbb{R}^3} \hat{\mu}_j(y) dy$. Rewriting $\frac{1}{|x-y|} = \int_0^1 \frac{d}{dt} \frac{1}{|x-ty|} dt + \frac{1}{|x|} = \int_0^1 \frac{(x-ty, y)}{|x-ty|^3} dt + \frac{1}{|x|}$ gives

$$\hat{\mu}_j(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} |x|^{-1} \Delta_{\mathbb{R}^3} \hat{\mu}_j(y) dy + \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_0^1 |x-ty|^{-3} (x-ty, y) \Delta_{\mathbb{R}^3} \hat{\mu}_j(y) dt dy$$

Set

$$\vartheta_j := \frac{1}{2\pi} \int_{\mathbb{R}^3} \Delta_{\mathbb{R}^3} \hat{\mu}_j(y) dy.$$

The second summand

$$\int_{\mathbb{R}^3} \int_0^1 |x-ty|^{-3} (x-ty, y) \Delta_{\mathbb{R}^3} \hat{\mu}_j(y) dt dy,$$

can be bounded by

$$C \int_{\mathbb{R}^3} \int_0^1 |x-ty|^{-2} |y|^{-3} dt dy$$

for self-dual connections and by

$$C \int_{\mathbb{R}^3} \int_0^1 |x-ty|^{-2} |y|^{-4} dt dy$$

for anti-self-dual connections. This integral can be broken into two integrals over the regions $\{y : |y| \geq \frac{|x|}{2}\}$ and $\{y : |y| < \frac{|x|}{2}\}$. The resulting terms are both $O\left(\frac{\ln(|x|)}{|x|^2}\right)$ in the anti-self-dual case and $O\left(\frac{1}{|x|^2}\right)$ in the self-dual case, and the result follows. The estimate of $d(\mu_j - \bar{\mu}_j(\infty))$ is similar.

For an anti-self-dual connection the form A in $\{w_j\}$ frame thus satisfies $A(\partial/\partial\tau) = -i \operatorname{diag}(\mu_j) = -i \frac{\lambda_j + m_j/r}{2V} + O(\ln r/r^2)$, with $m_j = l\vartheta_j + k\lambda_j$. The anti-self-duality relation implies that the connection itself has the form $A = A(\partial/\partial\tau)(d\tau + \omega) - \frac{1}{2}\pi_k^*(\eta) + O(\ln r/r^2)$ with $d\eta = *d\frac{m_j}{r}$. Thus, up to terms of order $1/r^2$ the local one-form η can be chosen to be $(m_j/k)\omega$.

Since $\{w_j\}$ trivializes the bundle over each circle fiber, the line bundle spanned by w_j descends to a line bundle over each sphere $S_R^2 \subset \mathbb{R}^3$ of a large radius R . The Chern number of this line bundle is exactly $\frac{1}{2\pi} \int_{S_R^2} d(-\frac{1}{2}\eta_j) = m_j$ and thus m_j is integer. \square

5 Asymptotic Decay of Harmonic Spinors

In order to recover the bow data from a generic instanton on TN_k , it is necessary to understand properties of the L^2 -kernel of the coupled Dirac operator. Of particular importance is the dimension of the kernel and the estimates of the decay rates of its members.

5.1 The Dirac Operator

Let A be an instanton connection with generic asymptotic holonomy on TN_k . As proved in the previous section, there is a gauge on E in which the instanton connection $d + A$ has $A\left(\frac{\partial}{\partial\tau}\right) = -\frac{i}{2V} \operatorname{diag}\left((\lambda_1 + \frac{m_1}{r}), \dots, (\lambda_n + \frac{m_n}{r})\right) + O\left(\frac{1}{r^2}\right)$. Generic asymptotic holonomy implies that the λ_j are pairwise distinct outside a compact set. For example, by a judicious choice of the frame at infinity they can be chosen to lie in the interval $[0, l)$. Denote by $S = S^+ \oplus S^-$ the spinor bundle of TN_k and its chiral decomposition under $Cl(Vdx^1dx^2dx^3d\tau)$. The connection A induces a coupled Dirac operator $D = D_A$ acting on $\Gamma(S \otimes E)$ with the chiral split:

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \quad (85)$$

where $D^+ : \Gamma(S^+ \otimes E) \rightarrow \Gamma(S^- \otimes E)$ and $D^- : \Gamma(S^- \otimes E) \rightarrow \Gamma(S^+ \otimes E)$.

5.2 Harmonic Spinors: the Fredholm Case

One of our objectives is to compute the L^2 -index of D_A . However, this operator is not always Fredholm. The following lemma characterizes which instanton connections produce Fredholm Dirac operators.

Lemma 19. *Given an instanton A over TN_k with generic asymptotic holonomy, if $\frac{\lambda_j}{l} \notin \mathbb{Z}$ for all j , then $D : H_1^2(S \otimes E) \rightarrow L^2(S \otimes E)$ is Fredholm.*

Proof. It is well known (see, e.g., [Ang93]) that D is Fredholm if and only if there is a compact set $K \subset \text{TN}_k$ and a constant C_K , such that $\|Dh\|_{L^2}^2 \geq C_K \|h\|_{L^2}^2$ for all $h \in H_1^2(S \otimes E)$ with $\text{supp}(h) \subset \text{TN}_k \setminus K$.

The scalar curvature of TN_k vanishes; so the Lichnerowicz formula reduces to $D^2 = \nabla^* \nabla + c(F_A)$. Here $c(F_A)$ denotes Clifford multiplication by the curvature form F_A . It follows that

$$\|Dh\|_{L^2}^2 = \|\nabla h\|_{L^2}^2 + \langle h, c(F)h \rangle_{L^2}.$$

From Theorem 14, we have $F_A = O(\frac{1}{r^2})$. Take K to be a ball of radius R centered at the origin. For h supported on the complement of K , we have $|\langle h, c(F_A)h \rangle| \leq C \|\frac{h}{r}\|^2 \leq \frac{C}{R^2} \|h\|^2$.

Outside K we can expand h in a Fourier series with respect to the periodic variable τ . This implies that

$$C\|\nabla h\|_{L^2}^2 \geq \|\nabla_{\partial_\tau} h\|_{L^2}^2 \geq \frac{1}{2} \left\| \min_{n \in \mathbb{Z}, j} \left| n + \frac{\lambda_j}{l} + \frac{m_j}{r} \right| h \right\|_{L^2}.$$

For R sufficiently large and $\frac{\lambda_j}{l} \notin \mathbb{Z}$, there exists $\beta > 0$ such that

$\frac{1}{2} \min_{n \in \mathbb{Z}, j} \left| n + \frac{\lambda_j}{l} + \frac{m_j}{r} \right|^2 > \beta^2$. Let $C_K = \beta^2 - \frac{C}{R^2}$. For R large enough, C_K is positive and the lemma follows. \square

Estimates implying Fredholmness often imply exponential decay of L^2 zero modes.

Proposition 20. *Let A be an instanton satisfying the hypotheses of the previous theorem and take $0 < \alpha < \inf \left\{ \left| \frac{\lambda_j}{l} + n \right|, n \in \mathbb{Z} \right\}$. Let $h \in \text{Ker}(D)$ with $e^{-\beta r} h \in L_2$ for some $\beta < \alpha$. Then $e^{br} h \in L_2$ for all $b < \alpha$ and h decays pointwise exponentially.*

Proof. Let $Dh = 0$, with $h \in L_2$. Let $\eta_n = e^{(b+\beta)r_n} e^{-\beta r}$, where $r_n = \min\{r, n\}$, and $b \geq \beta$. Then, as in our discussion of the curvature decay, we have

$$0 = \|D(\eta_n h)\|^2 - \|d\eta_n \otimes h\|^2 \geq \|\nabla(\eta_n h)\|^2 - b^2 \|\eta_n h\|^2 - C \|\eta_n r^{-1} h\|^2.$$

Definition 8 and the additional hypothesis on the λ_j imply that, for some compact set K ,

$$\|\nabla(\eta_n h)\|^2 \geq \|\alpha \eta_n h\|^2 - \int_K |\eta_n h|^2 dv.$$

Hence

$$0 \geq \alpha^2 \|\eta_n h\|^2 - b^2 \|\eta_n h\|^2 - \tilde{C} \|\eta_n r^{-1} h\|^2.$$

Choosing $b < \alpha$, we have

$$\tilde{C} \|\eta_n r^{-1} h\|^2 \geq (\alpha^2 - b^2) \|\eta_n h\|^2.$$

Taking the limit as $n \rightarrow \infty$, we see that $e^{br} h \in L_2$. We may now apply Moser iteration to deduce $e^{br} h$ is pointwise bounded. \square

5.3 Harmonic Spinors: the Non-Fredholm Case

The results of this section will not be used in the rest of this paper, but they will play a role in the third paper in this series. There the reconstruction of bow representations from generic instantons requires understanding the decay rates of L^2 -zero modes of D_A in some cases where the operator is not Fredholm. In this section we assume that $\lambda_1 = 0$ and $m_1 = 0$.

Theorem 21. *Let A be an instanton on TN_k with generic asymptotic holonomy and such that $\lambda_1 = 0$ and $m_1 = 0$. If D_A has an L^2 -zero mode h , then $|r^2 h|$ is pointwise bounded.*

Proof. An analogous argument to Proposition 10 proves that $|r^{3/2} h|$ is pointwise bounded and also that $|r^{5/2} \nabla h|$ is pointwise bounded. We use an orthonormal frame of the tangent bundle $\{\hat{e}_j\}_{j=1}^4$ of TN_k with $\hat{e}_4 = \sqrt{V} \partial_\tau$. We denote its coframe by $\{\hat{e}^j\}_{j=1}^4$, and let c^j denote the Clifford multiplication by \hat{e}^j .

Take r large and write

$$h = h_0 + h_1, \quad (86)$$

where h_0 is the projection of h onto the eigenspace of $\nabla_{\partial_\tau}^* \nabla_{\partial_\tau}$ with smallest eigenvalue; thus, $h_0 = \Pi_0 h$ and $h_1 = (1 - \Pi_0)h$, where

$$\Pi_0 := \frac{1}{2\pi i} \int_C (z - i\nabla_{\partial_\tau})^{-1} dz. \quad (87)$$

Here C is a small circle around 0. By Theorem 18,

$$|\nabla_{\partial_\tau} h_0| \leq \frac{c}{r^2} |h|. \quad (88)$$

Note that

$$\left[\nabla_{\frac{\partial}{\partial x^j}}, \Pi_0 \right] = \frac{1}{2\pi i} \int_C (z - i\nabla_{\partial_\tau})^{-1} iF \left(\frac{\partial}{\partial x^j}, \partial_\tau \right) (z - i\nabla_{\partial_\tau})^{-1} dz = O(r^{-2}). \quad (89)$$

Similarly, Corollary 15 implies

$$[c^a c^b F_{ab}, \Pi_0] = O(r^{-3}). \quad (90)$$

Lichnerowicz formula can be written as

$$\begin{aligned} 0 = \|D(\eta h)\|^2 - \|d\eta|h\|^2 &= \|\nabla(\eta h)\|^2 - \|d\eta|h\|^2 + \frac{1}{2} \langle c^a c^b F_{ab} \eta h_1, \eta h_1 \rangle \\ &\quad + \operatorname{Re} \langle c^a c^b F_{ab} \eta h_1, \eta h_0 \rangle + \frac{1}{2} \langle c^a c^b F_{ab} \eta h_0, \eta h_0 \rangle. \end{aligned} \quad (91)$$

Choosing $\eta = r$, we see $rh_1 \in L^2$. To estimate h_0 , we first bound the last curvature term in (91). The bound (90) implies

$$\left| \int_{S_{\bar{x}}^1} \frac{1}{2} (c^a c^b F_{ab} \eta h_0, \eta h_1) d\tau \right| \leq C \int_{S_{\bar{x}}^1} r^{-3} |\eta h_0| |\eta h_1| d\tau. \quad (92)$$

Using the anti-self-duality of A to express every curvature component in terms of $[\nabla_{\partial_r}, \nabla_{\partial_i}]$, for some i , integrating by parts, and then using (88), we have

$$\left| \int_{S_{\bar{x}}^1} (c(F)\eta h_0, \eta h_0) d\tau \right| \leq c \int_{S_{\bar{x}}^1} |\nabla(\eta h_0)| |r^{-2} \ln(r) \eta h_0| d\tau + c \left| \int_{S_{\bar{x}}^1} (c^j c^4 \nabla_{\hat{e}_j} \nabla_{\hat{e}_4} \eta h_0, \eta h_0) d\tau \right|, \quad (93)$$

where the index j is summed over $1, 2, 3$. To estimate the last term, we write

$$\int_{S_{\bar{x}}^1} (c^j c^4 \nabla_{\hat{e}_j} \nabla_{\hat{e}_4} \eta h_0, \eta h_0) d\tau = \int_{S_{\bar{x}}^1} (c^j c^4 \nabla_{\hat{e}_j} i\lambda \eta h_0, \eta h_0) d\tau. \quad (94)$$

By the definition of h_0 , the hypothesis on the λ_j 's and equation (83), $\lambda_1 = O(r^{-2} \ln r)$, and $|d\lambda| = O(r^{-3} \ln(r))$. Thus

$$\left| \int_{S_{\bar{x}}^1} (c(F)\eta h_0, \eta h_0) d\tau \right| \leq c' \int_{S_{\bar{x}}^1} |\nabla(\eta h_0)| |r^{-2} \ln(r) \eta h_0| d\tau + c'' \int_{S_{\bar{x}}^1} r^{-3} \ln(r) |\eta h_0|^2 d\tau. \quad (95)$$

Hence

$$\left| \int_{S_{\bar{x}}^1} (c(F)\eta h_0, \eta h_0) d\tau \right| \leq \int_{S_{\bar{x}}^1} \frac{\ln(r)}{r} |\nabla(\eta h_0)|^2 d\tau + C \int_{S_{\bar{x}}^1} r^{-3} \ln(r) |\eta h_0|^2 d\tau. \quad (96)$$

Now we prove that $r^a h \in L^2$ for $a < \frac{1}{2}$. Write

$$\nabla = \hat{e}^1 \otimes \nabla_{\hat{e}_1} + \nabla^0 = \hat{e}^1 \otimes \nabla_{\hat{e}_1} + \hat{e}_4 \otimes \nabla_{\hat{e}_4} + \nabla^{0,0},$$

where \hat{e}_1 is radial. Hardy's inequality gives

$$\|\nabla_{\hat{e}_1} \eta h\|^2 \geq \frac{1}{4} \left\| \frac{\eta h}{r} \right\|^2 - C_H \left\| \frac{\eta h}{r^{3/2}} \right\|^2.$$

Choosing $\eta = r_n^p r^{1/2}$, Bochner and Hardy inequalities give

$$\frac{1}{4} \left\| \frac{\eta h}{r} \right\|^2 + \|\nabla^0 \eta h\|^2 \leq \|d\eta h\|^2 + \langle c(F)\eta h, \eta h \rangle + C_H \left\| \frac{\eta h}{r^{3/2}} \right\|^2. \quad (97)$$

We first establish a lower bound for $\|r_n^p r^{1/2} \nabla^0 h\|^2$. Expanding

$$0 = \operatorname{Re}(Dh, c^1 h) r_n^{2p},$$

gives

$$\begin{aligned} -\operatorname{Re}(c^1 \nabla_{\hat{e}_1} h, c^1 h) r_n^{2p} &= \operatorname{Re} \left(\sum_{j>1} c^j \nabla_{\hat{e}_j} h, c^1 h \right) r_n^{2p} \\ &\leq \sqrt{2} |\nabla^{0,0} h| |h| r_n^{2p} + \operatorname{Re}(c^4 \nabla_{\hat{e}_4} h, c^1 h) r_n^{2p}. \end{aligned} \quad (98)$$

Hence

$$-\int_{S_{\vec{x}}^1} \operatorname{Re}(c^1 \nabla_{\hat{e}_1} h, c^1 h) r_n^{2p} \leq \sqrt{2} \int_{S_{\vec{x}}^1} |\nabla^{0,0} h| |h| r_n^{2p} + C \int_{S_{\vec{x}}^1} |h|^2 r_n^{2p} r^{-2} \ln(r), \quad (99)$$

by (88) and (89). Now

$$\operatorname{Re}(c^1 \nabla_{\hat{e}_1} h, c^1 h) r_n^{2p} = \frac{1}{2} \operatorname{Div}(|h|^2 r_n^{2p} \hat{e}_1) - (p \chi_{r \leq n} + 1) |h|^2 \frac{r_n^{2p}}{r} + \text{lower order}.$$

Here $\chi_{r \leq n}$ denotes the characteristic function of $\{r \leq n\}$.

Applying Cauchy-Schwartz inequality to (99) gives

$$\sqrt{2} \|r_n^p r^{1/2} \nabla^0 h\| \|r_n^p r^{-1/2} h\| \geq \int |h|^2 r_n^{2p} r^{-1} (pt_{n,p} + 1 - Cr^{-1} \ln(r)) d\nu, \quad (100)$$

where $t_{n,p} = \frac{\int |h|^2 r_n^{2p} r^{-1} \chi_{r \leq n} d\nu}{\int |h|^2 r_n^{2p} r^{-1} d\nu}$. Squaring both sides gives

$$\|\nabla^0 r_n^p r^{1/2} h\|^2 \geq \frac{(pt_{n,p} + 1)^2}{2} \|r_n^p r^{-1/2} h\|^2 - 2C(pt_{n,p} + 1) \int |h|^2 r_n^{2p} r^{-2} \ln(r) d\nu. \quad (101)$$

Combined with (97), (101) gives

$$\left(\frac{1}{4} + \frac{(pt_{n,p} + 1)^2}{2} - \left(\frac{1}{2} + p \right)^2 \right) \|r_n^p r^{-1/2} h\|^2 \leq \tilde{C} \left\| \frac{\eta h}{r^{3/2}} \right\|^2. \quad (102)$$

Letting $n \gg 0$, the coefficient on the left-hand-side is positive for $p < 1$, implying $\|r^{p-1/2} h\| < \infty$ for any $p < 1$.

The proof now continues as in the proof of Theorem 14, albeit simplified by the knowledge of the quadratic curvature decay. \square

6 The Index Theorem

So far we presumed that A is an instanton with generic asymptotic holonomy, so that $\exp(2\pi i \lambda_j / l)$ are pairwise distinct. For the rest of this paper in addition we also presume that it has no covariantly constant sections along a fiber at infinity, i.e. we adopt

Assumption 22. $\exp\left(2\pi i \frac{\lambda_j}{l}\right) \neq 1$ for all j .

As proved in Lemma 19, the corresponding Dirac operator $D_A : H_1^2(S \otimes E) \rightarrow L^2(S \otimes E)$ is Fredholm. The objective now is to compute its L^2 -index. The argument follows [Ste89] and [Ste93]:

1. In order to simplify the analysis, apply a conformal transformation to the original TN_k metric. This transformation does not change the L^2 -index of the Dirac operator. The new metric is asymptotically that of a circle bundle with shrinking fiber over an $\mathbb{R}_+ \times S^2$ base. Working with this new metric simplifies error estimates.

2. Express the index as a sum of terms involving the super-trace of the heat kernel e^{-tD^2} . The index can be written as a sum of two terms: the bulk and the asymptotic contribution. The bulk involves the classical Atiyah-Singer integrand, while the asymptotic contribution depends on the behavior at infinity of the instanton connection and on the rescaled metric.
3. Approximate the heat kernel by a parametrix, and show that the index can be computed from the parametrix. It follows, as in the compact case, that this approximation can be used to compute the bulk. It requires somewhat more work to show that the approximation can also be used to compute the asymptotic contribution. Our conformal change of the metric facilitates this step.
4. Finally, compute the index from the parametrix.

6.1 Index Preliminaries

The space TN_k is circle fibered over \mathbb{R}^3 and, moreover, outside a compact set it is an S^1 bundle. Fix an origin in the \mathbb{R}^3 base and let r denote the Euclidean distance (in \mathbb{R}^3) from this origin. Lift the function r to TN_k . Now we replace the original Taub-NUT metric g by a conformally equivalent metric $g' = Fg$, with $F = \frac{1}{Vr^2}$ for r large. The new metric takes the form

$$g' = \frac{1}{r^2} dr^2 + g_{S^2} + \frac{1}{V^2 r^2} (d\tau + \omega)^2 = dy^2 + g_{S^2} + e^{-2y} V^{-2} (d\tau + \omega)^2, \quad (103)$$

where $r = e^y$ and $\omega(\partial_\tau) = 0$. Outside of a compact set g' is a circle bundle over a cylinder $\mathbb{R}_y \times S^2$ with the S^1 -fibers shrinking rapidly as $y \rightarrow \infty$.

Proposition 23. *The Dirac operators associated to $g = g_{\text{TN}_k}$ and g' have the same L^2 -index.*

Proof. Near infinity, the conformal factor can be written as $e^{-2u} = \frac{1}{Vr^2}$ so, asymptotically, the corresponding Dirac operators are related by

$$D_{g'} = e^{\frac{5u}{2}} D_g e^{-\frac{3u}{2}}.$$

Define an injective map

$$\begin{aligned} U : \text{Ker}_{L^2}(D_g) &\rightarrow \text{Ker}_{L^2}(D_{g'}) \\ h &\mapsto e^{\frac{3u}{2}} h. \end{aligned} \quad (104)$$

Since the L^2 -solutions of $D_g \Psi = 0$ decay exponentially in r , the map takes L^2 to L^2 and is therefore well-defined. Now, let $\phi \in \text{Ker}_{L^2}(D_{g'})$, then $e^{-\frac{3u}{2}} \phi$ decays exponentially, since it belongs to $\text{Ker}(D_g)$, and $e^{-\beta r} e^{-\frac{3u}{2}} \phi \in L_2$ for all $\beta > 0$. Hence U is an isomorphism. \square

Henceforth we work in the conformally rescaled metric g' . In particular, $D := D_{g'}$, etc.

For a subset U of S^2 such that the S^1 -bundle over it is trivial. We choose a local oriented g' -orthonormal frame (e_1, e_2, e_3, e_4) with

$$e_1 = \partial_y, \quad e_2 = \bar{e}_2 - \omega(\bar{e}_2)\partial_\tau, \quad e_3 = \bar{e}_3 - \omega(\bar{e}_3)\partial_\tau, \quad e_4 = e^y V \partial_\tau, \quad (105)$$

where $\{\bar{e}_2, \bar{e}_3\}$ is a local oriented orthonormal frame on U lifted to TN_k via the above product structure. The coframe $e^1 = dy$, $e^2 = \pi^* \bar{e}^2$, $e^3 = \pi^* \bar{e}^3$ and $e^4 = e^{-y} V^{-1}(d\tau + \omega)$, where for $j = 2, 3$, $\{\bar{e}^j\}_{j=1}^3$ on $\mathbb{R} \times U$ is dual to $\{\bar{e}_j\}_{j=1}^3$, and π denotes the S^1 -bundle projection. Using standard spherical coordinates (ϕ, θ) on U , with polar angle $\phi \in [0, \pi]$ and azimuthal angle $\theta \in [0, 2\pi)$, and U not containing any poles, one can take $\bar{e}_2 = \partial_\phi$ and $\bar{e}_3 = \frac{1}{\sin \phi} \partial_\theta$. We denote from now on by c^j the Clifford multiplication by e^j .

The chirality operator⁴ that we introduced in the beginning of Section 5 is the Clifford multiplication by the volume form of g . In g' metric the chirality operator is $\gamma^5 = c(e^1 e^2 e^3 e^4) = c^1 c^2 c^3 c^4$. The two are related by the positive factor F^2 .

Lemma 24. *The L^2 -index of $D^+ : \Gamma(S^+ \otimes E) \rightarrow \Gamma(S^- \otimes E)$ is given by*

$$\begin{aligned} \text{ind}_{L^2} D^+ &= -\frac{\text{Rank}(E)}{192\pi^2} \int \text{tr}_{T(\text{TN}_k)} R \wedge R + \frac{1}{8\pi^2} \int \text{tr}_E F \wedge F \\ &+ \lim_{y \rightarrow \infty} \frac{1}{2} \int_{e^{-(2+\delta)y}}^\infty \int_{\partial M_y} \text{tr } c(\nu) \gamma^5 D e^{-tD^2}(x, x) d\nu_y dt, \end{aligned} \quad (106)$$

where M_y is an exhausting sequence of compact sets with smooth boundary, ν is the unit outward normal to ∂M_y , $d\nu_y$ is the induced volume form on ∂M_y , and $\delta > 0$ is a small number. We call the last summand the asymptotic contribution.

Proof. Let P denote the orthogonal projection onto $\text{Ker}(D)$, and let $p(x, x')$ denote the Schwarz kernel of P . Then

$$\text{ind}_{L^2} D^+ := \dim \text{Ker } D^+ - \dim \text{Ker } D^- = \text{Tr } \gamma^5 P = \int_M \text{tr } \gamma^5 p(x, x) d\nu,$$

where Tr denotes trace over the Hilbert space of L^2 sections, and tr denotes the pointwise trace of endomorphisms of $S \otimes E$.

Let $k(t, x, x')$ denote the Schwarz kernel of e^{-tD^2} . Observe that in the strong operator topology $P = \lim_{t \rightarrow \infty} e^{-tD^2}$. Hence $p(x, x') = \lim_{t \rightarrow \infty} k(t, x, x')$. This limit is not uniform in (x, x') , but is uniform on any compact subset. Hence,

$$\begin{aligned} \text{ind}_{L^2} D^+ &= \int_M \text{tr } \gamma^5 p(x, x) d\nu = \lim_{y \rightarrow \infty} \int_{M_y} \text{tr } \gamma^5 p(x, x) d\nu \\ &= \lim_{y \rightarrow \infty} \lim_{t \rightarrow \infty} \int_{M_y} \text{tr } \gamma^5 k(t, x, x) d\nu. \end{aligned}$$

⁴ It is important to note that the chirality operator in $2d$ -dimensional space that is often used in the literature (e.g. [BGV92, Chapter 3]) is defined as $i^d c^1 c^2 \dots c^{2d}$, which differs in sign from our definition of γ^5 .

Using the fundamental theorem of calculus and the divergence theorem, we rewrite this as⁵

$$\begin{aligned}
& \text{ind}_{L^2} D^+ \\
&= \lim_{y \rightarrow \infty} \left(\int_{e^{-(2+\delta)y}}^{\infty} \frac{d}{dt} \int_{M_y} \text{tr} \gamma^5 k(t, x, x) d\nu dt + \int_{M_y} \text{tr} \gamma^5 k(e^{-(2+\delta)y}, x, x) d\nu \right) \\
&= \lim_{y \rightarrow \infty} \left(\frac{1}{2} \int_{e^{-(2+\delta)y}}^{\infty} \int_{M_y} e_i \text{tr} c^i \gamma^5 Dk(t, x, x) d\nu dt + \int_{M_y} \text{tr} \gamma^5 k(e^{-(2+\delta)y}, x, x) d\nu \right) \\
&= \lim_{y \rightarrow \infty} \left(\frac{1}{2} \int_{e^{-(2+\delta)y}}^{\infty} \int_{\partial M_y} \text{tr} c(\nu) \gamma^5 Dk(t, x, x) d\nu dt + \int_{M_y} \text{tr} \gamma^5 k(e^{-(2+\delta)y}, x, x) d\nu \right),
\end{aligned}$$

We now analyze the summand

$$\lim_{y \rightarrow \infty} \int_{M_y} \text{tr} \gamma^5 k(e^{-(2+\delta)y}, x, x) d\nu. \quad (107)$$

Just as in the heat equation proof of the index theorem on compact manifolds, we iteratively construct an approximation $k_N(t, x, x')$ of the heat kernel, with $\text{tr} \gamma^5 k_N(t, x, x) dx$ converging to the Atiyah-Singer integrand as $t \rightarrow 0$. The approximation has the form

$$k_N(t, x, x') = \eta(x, x') h_t(x, x') \sum_{j=0}^N t^j \Theta_j(x, x'),$$

where $h_t(x, x') = \frac{1}{(4\pi t)^2} e^{-\frac{\text{dist}(x, x')^2}{4t}}$, and $\eta(x, x')$ is a cut-off function whose support is restricted to a small neighborhood \mathcal{N} of the diagonal, such, that for $(x, x') \in \mathcal{N}$, x lies in a normal coordinate neighborhood of x' and x' lies in a normal coordinate neighborhood of x . We assume also that η is identically one in a smaller neighborhood of the diagonal. By Duhamel's principle, the error terms in this approximation are controlled by

$$\epsilon_N := \left(\frac{\partial}{\partial t} + D^2 \right) k_N.$$

For injectivity radius small relative to t , the norm of ϵ_N is dominated by norms of the terms involving derivatives of the cutoff functions η . Denote by rad the injectivity radius, then the η derivatives are supported in a region $\text{dist}(x, x') >$

⁵ The second equality in this expression follows from $\int_{M_y} \frac{d}{dt} \text{tr} \gamma^5 k(x, x')|_{x=x'} d\nu = - \int_{M_y} \text{tr} \gamma^5 (D^2 k)(x, x')|_{x=x'} d\nu = - \int_{M_y} \text{tr} \gamma^5 (D_x k D_{x'})|_{x=x'} d\nu = - \frac{1}{2} \int_{M_y} \text{tr} \gamma^5 ((D_x + D_{x'}) D_x k)|_{x=x'} d\nu = \frac{1}{2} \int_{M_y} e_i \text{tr} c^i \gamma^5 (Dk(x, x)) d\nu.$

C_{rad} for some $0 < C < 1$. For the g' metric, rad decays as e^{-y} . Also, for $(x, x') \in \text{support } d\eta$,

$$h_t(x, x') \leq e^{-\hat{C} \frac{\text{rad}^2}{2t}},$$

for another constant $\hat{C} > 0$. Taking $t \leq e^{-(2+\delta)y}$ implies then that the error terms associated to derivatives of η decay as $e^{-e^{\delta y}} \text{Pol}(e^y)$, where $\text{Pol}(e^y)$ is a polynomial expression in e^y . The error terms not involving derivatives of η can be bounded by high powers of t , exactly as in the compact case. It follows, as in the compact case, that (107) equals $-\int \text{ch}(E) \wedge \hat{A}(\text{TN}_k)$. The $\hat{A} = 1 - \frac{1}{24}p_1 \dots = 1 + \frac{1}{192\pi^2} \text{tr} R \wedge R$, where p_1 is the first Pontryagin class. On the other hand, the Chern character equals $1 + \frac{i}{2\pi} \text{tr} F - \frac{1}{8\pi^2} \text{tr} F_A \wedge F_A$. Therefore, (107) equals $-\frac{\text{Rank}(E)}{192\pi^2} \int \text{tr} R \wedge R + \frac{1}{8\pi^2} \int \text{tr} F_A \wedge F_A$. \square

Lemma 25. *On TN_k the Pontryagin class equals*

$$\frac{1}{192\pi^2} \int \text{tr} R \wedge R = \frac{k}{12}. \quad (108)$$

Proof. This is computed in [Haw77] and can also be established by direct calculation. We have as in [NS96] that $\text{tr} R \wedge R = \frac{1}{2}(\Delta \Delta V^{-1}) dx^1 \wedge dx^2 \wedge dx^3 \wedge d\tau$, where x^j are coordinates on \mathbb{R}^3 and $\Delta = \partial_{x^1}^2 + \partial_{x^2}^2 + \partial_{x^3}^2$. Since V is harmonic, $\Delta V^{-1} = 2 \frac{|\nabla V|^2}{V^3}$, implying that $|R|$ decays cubically at infinity and that $\Delta V^{-1} = \frac{2}{r_\sigma} + O(r_\sigma^0)$ near each center ν_σ . Applying Stokes theorem, the computation of the Pontryagin number reduces to

$$-\sum_{\sigma} \frac{1}{48\pi^2} \int_{S_{\nu_\sigma}^2} \nabla \frac{|\nabla V|^2}{V^3} \cdot d\text{Vol}_{S_{\nu_\sigma}^2},$$

where $S_{\nu_\sigma}^2$ denotes a small sphere centered at $\nu_\sigma \in \mathbb{R}^3$. (Cubic decay of $|R|$ implies that there is no contribution from infinity.) The integral over each center contributes $\frac{1}{12}$.

This Lemma also follows from a more general statement [AL13, Sec.5]: $\int_M |R|^2 d\nu = 8\pi^2 \chi(M)$, for any gravitational instanton obtained as a limit of $M = \bar{M} \setminus \Sigma$, a compliment of Σ within a compact smooth four-manifold with edge-cone singularity \bar{M} along Σ . In our case, R is anti-self-dual, thus $\text{tr} R \wedge R = -\text{tr} R \wedge *R = 2|R|^2$. Since TN_k is contractible to a bouquet of $k-1$ spheres $\bigvee_{i=1}^k S^2$, its Euler characteristic is $\chi(\text{TN}_k) = k$. \square

6.2 Approximation of the Heat Kernel

Equation (106) for the index of D^+ involves two summands: the Atiyah-Singer integrand and the asymptotic contribution. The latter is more subtle; hence we will give more details of the analysis of this term. We first specify an iterative semilocal approximation to the heat kernel. We then prove that substituting the approximation for the exact kernel computes the asymptotic contribution to the index.

Consider an open neighborhood in TN_k of the form

$$\mathcal{U} = (R, \infty) \times U \times S^1,$$

In the following, working in the coordinates we introduced in Section 6.1, let x denote coordinates (y, ϕ, θ, τ) on $(R, \infty) \times U \times S^1$, and let $b = (b_1, b_2, b_3) = (y, \phi, \theta)$ so that $x = (b, \tau)$.

For a suitable contour C surrounding $[0, \infty) \subset \mathbb{R} \subset \mathbb{C}$ oriented counterclockwise, we write

$$e^{-tD^2} = \frac{-1}{2\pi i} \int_C e^{-t\lambda} (D^2 - \lambda)^{-1} d\lambda.$$

Hence, an approximation of $(D^2 - \lambda)^{-1}$ yields an approximation of the heat kernel.

To approximate the resolvent on \mathcal{U} we use the continuous Fourier transform in the base variables and the discrete Fourier expansion in the τ -fibers. Let v denote coordinates dual to the b variables. Write $u = (v, \frac{k}{4\pi}) = (v_1, v_2, v_3, \frac{k}{4\pi})$, where $k \in \mathbb{Z}$ labels⁶ the discrete Fourier modes in the S^1 -fiber. We shall make use of the compact notation

$$\int \dots du := \sum_{k \in \mathbb{Z}} \int \dots dv.$$

Now iteratively construct an approximation of $(D^2 - \lambda)^{-1}$ of the form

$$\begin{aligned} & \sum_{j=0}^{\infty} \int e^{2\pi i(x-x')u} \sigma^{-j-1} q_j \psi(x, x') du \\ &= \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^3} e^{2\pi i(b-b')v} e^{\frac{i}{2}(\tau-\tau')k} \sigma^{-j-1} q_j \psi(x, x') dv, \end{aligned} \quad (109)$$

where σ is constructed below from the symbol of D^2 , ψ is a fiber isomorphism specified below, and the q_j are defined inductively. Observe that

$$\int e^{2\pi i(y-y')v_1} e^{2\pi i(\phi-\phi')v_2} e^{2\pi i(\theta-\theta')v_3} dv_1 dv_2 dv_3$$

represents the (distributional) integral kernel of the delta distribution with respect to the form $dy \wedge d\phi \wedge d\theta$.

We now explain each term in (109). On \mathcal{U} , $\psi(x, x') \in \text{Hom}(S_{x'} \otimes E_{x'}, S_x \otimes E_x)$ is a frame dependent identification of the fibers of the bundle $S \otimes E$ at x' and at x (see [CS15]). The local frame $\{e_j\}_{j=1}^4$, defines a section of the bundle of oriented orthonormal frames. We lift this to a local section of the principal spin bundle and use it to define a local frame $\{f_a\}_{a=1}^4$ for the spin bundle. Define $\psi^S(x, x') = f_a(x) \otimes f_a^*(x')$ where $\{f_a^*\}$ denotes the dual coframe. Take a local

⁶ Note the font difference between the Fourier label k and the number of centers k of TN_k . We also recall that τ has period 4π .

unitary frame $\{s_l\}$ of E on \mathcal{U} and define $\psi^E(x, x') = s_l(x) \otimes s_l^*(x')$, where again $\{s_l^*\}$ denotes the dual frame. Now set $\psi = \psi^S \otimes \psi^E$. Observe that

$$\psi_S^{-1} \nabla_{e_j}^S \psi^S(x, x') = \Gamma_j^S(x), \quad \psi_E^{-1} \nabla_{e_j}^E \psi^E(x, x') = A_j^E(x), \quad (110)$$

where $e^j \otimes \Gamma_j^S(x)$ and $e^j \otimes A_j^E$ are the connection one forms for the given frames. For the computations below, we fix a frame at infinity of E , such that the generic instanton has the asymptotics of Theorem 18, and parallel translate it along the y direction to a frame on $(R, \infty) \times U \times S^1$.

Write

$$e^{-\frac{i}{2}\tau k} e^{-2\pi i b \cdot v} D e^{2\pi i b \cdot v} e^{\frac{i}{2}\tau k} = D + c(du), \quad (111)$$

then

$$c(du) = 2\pi i v_1 c^1 + (2\pi i v_2 - \omega(\partial_\phi) \frac{i}{2} k) c^2 + \frac{2\pi i v_3 - \omega(\partial_\theta) \frac{i}{2} k}{\sin \phi} c^3 + e^y V \frac{i}{2} k c^4 \quad (112)$$

In order to simplify certain Poisson summations later on, we modify these relations slightly. Denote by Λ the limiting value at infinity of $2iA(\partial_\tau)$. In this frame Λ is a real diagonal matrix: $\Lambda = \text{diag}(\lambda_j/l)$.

Write analogously to (111)

$$e^{-\frac{i}{2}\tau k} e^{-2\pi i b \cdot v} D e^{2\pi i b \cdot v} e^{\frac{i}{2}\tau k} = \hat{D} + c(\delta u), \quad (113)$$

with

$$\begin{aligned} c(\delta u) &= 2\pi i v_1 c^1 + (2\pi i v_2 - \omega(\partial_\phi) \frac{i}{2} (k - \Lambda)) c^2 \\ &\quad + \frac{1}{\sin \phi} (2\pi i v_3 - \omega(\partial_\theta) \frac{i}{2} (k - \Lambda)) c^3 + e^y V \frac{i}{2} (k - \Lambda) c^4 \\ &= 2\pi i v_1 c^1 + 2\pi i v_2 c^2 + \frac{1}{\sin \phi} 2\pi i v_3 c^3 + e^y V \frac{i}{2} (k - \Lambda) c^4 - c(\omega) \frac{i}{2} (k - \Lambda), \end{aligned} \quad (114)$$

and \hat{D} is defined by equation (113). It follows that

$$\begin{aligned} c(\delta u)^2 &= (2\pi v_1)^2 + \frac{1}{4} e^{2y} V^2 (k - \Lambda)^2 \\ &\quad + \left(2\pi v_2 - \frac{\omega(\partial_\phi)}{2} (k - \Lambda) \right)^2 + \frac{1}{\sin^2 \phi} \left(2\pi v_3 - \frac{\omega(\partial_\theta)}{2} (k - \Lambda) \right)^2. \end{aligned} \quad (115)$$

For $\lambda \in \mathbb{C}$, set

$$\sigma = c(\delta u)^2 - \lambda. \quad (116)$$

Assumption 22 implies that no diagonal entry of $(k - \Lambda)$ is zero, therefore σ^{-1} decays as e^{-2y} for $\lambda \in C$.

Let us introduce differential operators $L_j = \sigma^j L \sigma^{-j}$, with $L = \hat{D}^2 + \{\hat{D}, c(\delta u)\}$ so that

$$\sigma^j e^{-2\pi i(x-x')u} (D^2 - \lambda) e^{2\pi i(x-x')u} \sigma^{-j} = L_j + \sigma,$$

Thus

$$\begin{aligned}
(D^2 - \lambda) \int e^{2\pi i(x-x') \cdot u} \sigma^{-j-1} q_j \psi(x, x') du &= \\
\int e^{2\pi i(x-x') \cdot u} \sigma^{-j-1} (\sigma + L_{j+1}) q_j \psi(x, x') du &= \\
\int e^{2\pi i(x-x') \cdot u} \sigma^{-j} q_j \psi(x, x') du + \int e^{2\pi i(x-x') \cdot u} \sigma^{-j-1} L_{j+1} q_j \psi(x, x') du.
\end{aligned}$$

Set $q_0(x, x') \in \text{End}(S_x \otimes E_x)$ to be the identity, and define $q_j(x, x') \in \text{End}(S_x \otimes E_x)$ by setting $q_{j+1} \psi = -L_{j+1} q_j \psi$, so that

$$\begin{aligned}
(D^2 - \lambda) \sum_{j=0}^N \int e^{2\pi i(x-x') \cdot u} \sigma^{-j-1} q_j \psi(x, x') du &= \\
\int e^{2\pi i(x-x') \cdot u} \psi(x, x') du + \int e^{2\pi i(x-x') \cdot u} \sigma^{-N-1} L_N q_N \psi(x, x') du &= \\
= I + \int e^{2\pi i(x-x') \cdot u} \sigma^{-N-1} L_N q_N \psi(x, x') du.
\end{aligned}$$

To construct a global approximate resolvent, we introduce partitions of unity and cutoff functions required to localize the kernels to a neighborhood of the diagonal where our coordinates are defined. Fix a locally finite open cover $\{\tilde{U}_\alpha = (R_\alpha - 1, R_\alpha + 1) \times U_\alpha\}_\alpha$ of $(R, \infty) \times S^2$ so that the S^1 bundle is trivial over each U_α and the U_α are geodesic balls of radius $10e^{-\frac{3}{4}R_\alpha}$. Choose functions $\{\xi_\alpha\}_\alpha$ and $\{\tilde{\xi}_\alpha\}_\alpha$ with $|d\xi_\alpha| < 10e^{\frac{3}{4}R_\alpha}$ and $|d\tilde{\xi}_\alpha| < 10e^{\frac{3}{4}R_\alpha}$, and such that $\zeta_\alpha := \xi_\alpha \circ \pi_k$ and $\tilde{\zeta}_\alpha := \tilde{\xi}_\alpha \circ \pi_k$ satisfy the following conditions.

1. $\{\zeta_\alpha\}_\alpha$ is a partition of unity subordinate to $\{\pi_k^{-1}(\tilde{U}_\alpha)\}$.
2. ζ_α is supported on the set $\tilde{\zeta}_\alpha = 1$.
3. For R_α large, $d(x_1, x_2) > e^{-\frac{3}{4}R_\alpha}$, for all (x_1, x_2) with $\zeta_\alpha(x_1)d\tilde{\zeta}_\alpha(x_2) \neq 0$.
4. $d(x_1, x_2) < 10e^{-\frac{3}{4}R_\alpha}$, for all $x_1, x_2 \in \text{supp } \zeta_\alpha$.

In this construction, the function $e^{-\frac{3}{4}R_\alpha}$ can be replaced by e^{-CR_α} for any $C \in (0, 1)$. If $C \geq 1$, then stationary phase arguments no longer imply the error terms associated with derivatives of the cutoffs are rapidly decreasing. The larger we choose C , however, the better our subsequent bounds on connection matrices. Fix a point $o_\alpha \in U_\alpha \times \{1\}$ and choose a frame $\{f_a^\alpha\}_a$ for E on $\pi_k^{-1}(U_\alpha)$ as follows. The frame is radially covariant constant on $\tilde{U}_\alpha \times \{1\}$ and is extended to $\pi_k^{-1}(U_\alpha)$ so that its coefficients in the $\{w_j\}_j$ frame constructed in the first paragraph of section 4 are constant. With this choice of frame, the connection matrices satisfy

$$A_j = \frac{1}{2} F(r_\alpha \frac{\partial}{\partial r_\alpha}, e_j) + O(r_\alpha^3), \quad (117)$$

where r_α denotes radial distance from o_α . (See, for example, [CS15, Lemma 3.18].) Hence

$$|A_j| = O(e^{-\frac{3y}{4}}), \text{ for all } j < 4, \quad (118)$$

and decomposing $A = A^Z + A^B$ into summands which commute with A_4 and those orthogonal to the centralizer of A_4 , we have by Corollary 17,

$$|A_j^B| = O(e^{-\frac{7y}{4}}), \text{ for all } j. \quad (119)$$

Then our approximation to the heat kernel for large y is the operator K_t^N with integral kernel given by

$$k_t^N(x, x') := \sum_{\alpha} \tilde{\zeta}_{\alpha}(x) \zeta_{\alpha}(x') \sum_{j=0}^N \frac{-1}{2\pi i} \int_C e^{-t\lambda} \int e^{2\pi i(x-x')u} \sigma^{-j-1} q_{\alpha,j} \psi_{\alpha}(x, x') du d\lambda, \quad (120)$$

where ψ_{α} and $q_{\alpha,j}$ are constructed as before, but with the geometric data, such as local frames for E , dependent on the set U_{α} . For N a large positive integer, let

$$q_{\lambda}^N(x, x') := \sum_{\alpha} \tilde{\zeta}_{\alpha}(x) \zeta_{\alpha}(x') q_{\alpha,\lambda}^N(x, x'), \quad (121)$$

denote the approximate resolvent, where

$$q_{\alpha,\lambda}^N(x, x') := \sum_{j=0}^N \int e^{2\pi i(x-x')u} \sigma^{-j-1} q_{\alpha,j} \psi_{\alpha}(x, x') du, \quad (122)$$

and let Q_{λ}^N denote the operator with Schwartz kernel q_{λ}^N .

Lemma 26. *For any $t > 0$ and N large,*

$$|\text{Tr} \gamma^5 \eta e^{-tD^2} - \text{Tr} \gamma^5 \eta K_t^N| = O\left(t^N \int \eta(y) \text{tr} e^{-te^{2y}(k-\Lambda)^2} dy\right).$$

where η is a cutoff function supported in a neighborhood of ∞ and identically 1 in a smaller neighborhood of ∞ .

Proof. For $z \in \mathbb{C}$ with $\text{Im}(z) \neq 0$ and any $\phi \in H_1^2(S \otimes E)$, one has $\text{Im}((D^2 - z)\phi, \phi) = -\text{Im}(z)\|\phi\|^2$. Thus, $\|(D^2 - z)\phi\|_{L^2} \geq |\text{Im}(z)|\|\phi\|$. This implies that $D^2 - z$ is injective with closed range. The corresponding inverse satisfies, in the operator norm, $|(D^2 - z)^{-1}|_{\text{op}} \leq |\text{Im}(z)|^{-1}$.

Consider a counterclockwise oriented curve C_t surrounding the spectrum of D^2 defined as follows: C_t is the union of a semicircle $\{z : |z| = 1/t, \text{Re} z \leq 0\}$ and two horizontal half-lines $y = \pm \frac{1}{t}$, $x \geq 0$. Observe that $|e^{-t\lambda}| \leq e$ for any $\lambda \in C_t$. Moreover, for all $\lambda \in C_t$,

$$|(D^2 - \lambda)^{-1}|_{\text{op}} \leq t.$$

Write

$$p_t(x, x') = \frac{i}{2\pi} \int_{C_t} e^{-t\lambda} r_{\lambda}(x, x') d\lambda,$$

where $r_\lambda(x, x')$ is the Schwartz kernel for $(D^2 - \lambda)^{-1}$.

Let $\epsilon_\lambda^N := I - (D^2 - \lambda)Q_\lambda^N$. Then

$$\begin{aligned} e^{-tD^2} - K_t^N &= \frac{i}{2\pi} \int_{C_t} e^{-t\lambda} ((D^2 - \lambda)^{-1} - Q_\lambda^N) d\lambda \\ &= \frac{i}{2\pi} \int_{C_t} e^{-t\lambda} (D^2 - \lambda)^{-1} \epsilon_\lambda^N d\lambda. \end{aligned}$$

Consider now

$$\begin{aligned} \text{Tr} \gamma^5 \eta (e^{-tD^2} - K_t^N) &= \text{Tr} \gamma^5 \eta \frac{i}{2\pi} \int_{C_t} e^{-t\lambda} (D^2 - \lambda)^{-1} \epsilon_\lambda^N d\lambda \\ &= \text{Tr} \gamma^5 \eta \frac{i}{2\pi} \int_{C_t} e^{-t\lambda} Q_\lambda^N \epsilon_\lambda^N d\lambda \\ &\quad + \text{Tr} \gamma^5 \eta \frac{i}{2\pi} \int_{C_t} e^{-t\lambda} (D^2 - \lambda)^{-1} (\epsilon_\lambda^N)^2 d\lambda \end{aligned}$$

Hence

$$\begin{aligned} |\text{Tr} \gamma^5 \eta (e^{-tD^2} - K_t^N)| &\leq \left| \text{Tr} \gamma^5 \eta \frac{1}{2\pi} \int_{C_t} e^{-t\lambda} Q_\lambda^N \epsilon_\lambda^N d\lambda \right| \\ &\quad + \frac{1}{2\pi} \int_{C_t} e^{-t\lambda} |t| |\sqrt{\eta} \epsilon_\lambda^N|_{HS} |\epsilon_\lambda^N \sqrt{\eta}|_{HS} d|\lambda|. \end{aligned} \quad (123)$$

Here $|\cdot|_{HS}$ denotes the Hilbert-Schmidt norm. Expand

$$\epsilon_\lambda^N = \sum_\alpha \left[D^2, \tilde{\zeta}_\alpha(x) \right] \zeta_\alpha(x') q_{\alpha,\lambda}^N + \tilde{\epsilon}_\lambda^N.$$

The term $\sum_\alpha [D^2, \tilde{\zeta}_\alpha(x)] \zeta_\alpha(x') q_{\alpha,\lambda}^N$ is supported where $d(x, x') > e^{-\frac{3y}{4}}$, but $(\frac{\partial}{\partial v})^J \sigma^{-1} = O(e^{-(|J|+2)y})$. By stationary phase approximation, we conclude that the trace of this sum is rapidly decreasing as $y \rightarrow \infty$. By construction,

$$\tilde{\epsilon}_\lambda^N = \sum_\alpha \tilde{\zeta}_\alpha(x) \zeta_\alpha(x') \int e^{2\pi i(x-x')u} \sigma^{-N-1} L_N q_N \psi(x, x') du.$$

The corresponding integrals involving $\tilde{\epsilon}_\lambda^N$ on the right hand side of (123) give terms of order $O(t^{N-2} \exp(-te^{2y}(\mathbf{k} - \Lambda)^2))$. \square

A similar result holds for $De^{-tD^2} - DK_t^N$. Hence, we can use the approximate heat kernel K_t^N to compute the asymptotic contribution to the index. We are thus left to compute

$$\begin{aligned} \lim_{y \rightarrow \infty} \frac{1}{2} \int_{e^{-(2+\delta)y}}^{\infty} \int_{\partial M_y} \text{trc}(\nu) \gamma^5 D e^{-tD^2}(x, x) d\nu_x dt &= \\ \lim_{y \rightarrow \infty} \frac{1}{2} \int_{e^{-(2+\delta)y}}^{\infty} \int_{\partial M_y} \frac{-1}{2\pi i} \int_C \int \sum_{j=0}^N e^{-t\lambda} \text{trc}(\nu) \gamma^5 (\hat{D} + c(\delta u)) \sigma^{-j-1} q_j \psi|_{x=x'} du d\lambda d\nu_x dt. \end{aligned} \quad (124)$$

In what follows, set $M_s = y^{-1}([0, s])$. Then the unit normal vector field to ∂M_s is $\nu = e_1 = \partial_y$. The following section is devoted to simplifying (124).

6.3 The Grading

The expression (124) contains an enormous number of summands. In this section we show that only those summands corresponding to q_0 and q_1 contribute to the index formula. First we rescale our variables letting $\tilde{u} = (v, e^{y \frac{k}{4\pi}})$. Expand

$$q_j = q_{j,B,m} \tilde{u}^B \sigma^{-m}.$$

Here the $q_{j,B,m}$ are independent of \tilde{u} , and we sum over the repeated indices. In the expansion above, $B = (b_1, b_2, b_3, b_4)$ denotes a multiindex and $\tilde{u}^B = v^{b_1} v_2^{b_2} v_3^{b_3} (e^{y \frac{k}{4\pi}})^{b_4}$ is the corresponding monomial in the \tilde{u} -variables. Here there is a slight abuse of notation. Since σ is not a scalar multiple of the identity, we should write instead

$$q_j = q_{j,B,m_1,1} \sigma^{-m_1} q_{j,B,m_2,2} \sigma^{-m_2} \cdots q_{j,B,m_l,l} \sigma^{-m_l} \tilde{u}^B,$$

with $m_1 + \cdots + m_l = m$. It is easy to check, however, that since $A_j^B = O(e^{-\frac{7y}{4}})$, the commutator terms do not contribute to the trace. Hence we will use the more compact notation, suppressing the commutator terms. Similarly, we suppress terms in our notation which do not preserve Fourier mode. By Proposition 17, these terms decay exponentially and do not contribute. We have the following simple lemma.

Lemma 27. *For $|B| > 2m + \min\{j, 2j - m\}$, $q_{j,B,m} = 0$.*

Proof. Recalling that

$$\begin{aligned} q_j &= (-1)^j L_j \circ L_{j-1} \circ \cdots \circ L_1 \text{Id} \\ &= (-1)^j \sigma^j \circ L \circ \sigma^{-1} \circ L \circ \sigma^{-1} \cdots L(\sigma^{-1} \text{Id}), \end{aligned}$$

the highest power of σ in this expansion is zero. Differentiation of σ^{-1} in spatial directions does not increase the asymptotic \tilde{u} degree of homogeneity. Hence, application of L can only increase the asymptotic \tilde{u} homogeneity by one, (arising from the summand in L homogeneous of degree one in \tilde{u} .) Therefore, $\tilde{u}^B \sigma^{-m}$ must have asymptotic \tilde{u} homogeneity at most j , and this implies that $|B| - 2m \leq j$. Moreover, terms with $m = j + m_0$, $m_0 > 0$ arise in the iterative construction in which at m_0 steps in the iteration, σ^{-1} is differentiated twice and therefore involves only terms in $L'_i s$ homogeneous of degree 0 at that step. This gives the sharper estimate. \square

The q_j decomposition yields a corresponding decomposition of k_t^N .

$$\begin{aligned} k_t^N(x, x') &= \sum_{j=0}^N \frac{-1}{2\pi i} \int_C e^{-t\lambda} \int e^{2\pi i(x-x')u} \sigma^{-j-1-m} q_{j,B,m} \tilde{u}^B \psi(x, x') du d\lambda \\ &= \sum_{j=0}^N \int e^{2\pi i(x-x')u} e^{-tc(\delta u)^2} \frac{(-t)^{m+j}}{(m+j)!} q_{j,B,m} \tilde{u}^B \psi(x, x') du. \end{aligned}$$

Recalling expression (111) for $c(\delta u)^2$ and changing variables

$$\eta_1 = t^{1/2}v_1, \quad \eta_2 = t^{1/2}(v_2 - \omega(\partial_\phi)\frac{(\mathbf{k} - \Lambda)}{4\pi}), \quad \eta_3 = \frac{t^{1/2}}{\sin\phi}(v_3 - \omega(\partial_\theta)\frac{(\mathbf{k} - \Lambda)}{4\pi}),$$

changes $k_t^N(x, x')$ to

$$\sum_{j,\mathbf{k}} e^{\frac{i}{2}(\tau-\tau')\mathbf{k}} e^{-te^{2y}V^2\frac{1}{4}(\mathbf{k}-\Lambda)^2} \times \int e^{2\pi i(b-b')t^{-\frac{1}{2}}\eta} e^{-4\pi^2|\eta|^2} \frac{(-t)^{m+j}}{(m+j)!} q_{j,B,m} p_B(t^{-1/2}\eta, e^y\mathbf{k}) \psi(x, x') t^{-3/2} \sin\phi d\eta.$$

Here $p_B(t^{-1/2}\eta, e^y\mathbf{k})$ is the polynomial of degree B obtained from \tilde{u}^B by the change of variables.

Hence,

$$(Dk_t^N)(x, x) = \sum_{j,\mathbf{k}} \int \left(\sum_{a<3} c_a(e_a + 2\pi\frac{i\eta_a}{t^{1/2}} + A_a + \Gamma_a^S) + c_4 e^y V(i(\frac{\mathbf{k}}{2} - iA_4) + \Gamma_4^S) \right) \times e^{-te^{2y}V^2\frac{1}{4}(\mathbf{k}-\Lambda)^2} e^{-4\pi^2|\eta|^2} \frac{(-t)^{m+j-\frac{3}{2}}}{(m+j)!} q_{j,B,m} p_B(t^{-\frac{1}{2}}\eta, e^y\mathbf{k}) \sin\phi d\eta. \quad (125)$$

Observe, that A_a and Γ_a act on the q_j via the adjoint action, as the q_j are sections of the adjoint bundle. However, differentiating ψ brings additional connection matrices on the right hand side of the q_j . In other words:

$$\nabla_{e_m}(q_j\psi) = (\nabla_{e_m}q_j)\psi + q_j\nabla_{e_m}\psi = e_m(q_j\psi) + (\Gamma_m + A_m)q_j\psi,$$

where, in the expression $(\Gamma_m + A_m)q_j\psi$, Γ_m is acting via Clifford multiplication and A_m is composition with an element of $\text{End}(E)$ rather than commutator. Therefore, we may suppress the derivatives of ψ , if we understand the connection matrices to act in the fundamental - not the adjoint representation. We did so in (125), and henceforth that will be our convention.

We next define a weight on the summands of the asymptotic expansion. These weights simplify the determination of which terms contribute to the index.

Definition 28 (Clifford Degree). Consider a partial differential operator of the form $\sum_{|I|=j} c(e^I)p_I$, with a multi-index $I = \{i_1, \dots, i_{|J|}\}$, $e^I = e^{i_1} \wedge \dots \wedge e^{i_{|J|}}$, and p_I a partial differential operator that commutes with c^j for all j . We say such an operator has *Clifford degree* j .

For α a p -form on a four-manifold $\text{tr}\gamma^5 c(\alpha) = 0$, unless $p = 4$. Thus, the trace of any summand in $c^1 Dk_t^N$ is zero unless it has Clifford degree 4.

We further filter the space of partial differential operators.

Definition 29 (Filtration on Differential Operators). A partial differential operator has weight $w \leq |I| + |J| + l - 2m$ if it can be written in the form $\sum_{|I|=j} c(e^I)\sigma^{-m-1}v^J e^{ly} p_{IJlm}$, where p_{IJlm} is a partial differential operator with coefficients (in our orthonormal frame) bounded in C^k for all k , asymptotically homogeneous of degree 0 in \tilde{u} .

This filtration respects composition: if p_1 and p_2 have weight w_1 and w_2 respectively, then $p_1 \circ p_2$ has weight less than or equal to $w_1 + w_2$.

Lemma 30. For $\chi \geq 0$, $p > -\frac{1}{2}$ and $a \notin \mathbb{Z}$,

$$\sum_{k \in \mathbb{Z}} \int_{\chi}^{\infty} t^p e^{-te^{2y} V^2 \frac{1}{4}(k+a)^2} dt \leq C_p e^{-2(p+1)y},$$

for some $C_p > 0$. For $0 < \delta < 1$ and b odd

$$\sum_{k \in \mathbb{Z}} \int_{e^{-(2+\delta)y}}^{\infty} t^q \left[e^y V \frac{1}{2}(k+a) \right]^b e^{-te^{2y} V^2 \frac{1}{4}(k+a)^2} dt = O\left(e^{-(2q+2-b)y}\right),$$

For b even we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \int_{e^{-(2+\delta)y}}^{\infty} t^q \left[e^y V \frac{1}{2}(k+a) \right]^b e^{-te^{2y} V^2 \frac{1}{4}(k+a)^2} dt \\ = O\left(e^{-(2q+2-b)y}\right) + O\left(e^{\frac{\delta}{2}y - (2q+2-b)(1+\frac{\delta}{2})y}\right). \end{aligned}$$

Proof. We break the integration interval into two pieces $[e^{-(2+\delta)y}, e^{-2y}]$ and $[e^{-2y}, \infty)$

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \int_{e^{-2y}}^{\infty} t^q \left[e^y V \frac{1}{2}(k+a) \right]^b e^{-te^{2y} V^2 \frac{1}{4}(k+a)^2} dt \\ = \sum_{k \in \mathbb{Z}} e^{-(2+2q-b)y} \int_1^{\infty} t^q \left[V \frac{1}{2}(k+a) \right]^b e^{-tV^2 \frac{1}{4}(k+a)^2} dt = O\left(e^{-(2+2q-b)y}\right). \end{aligned}$$

To estimate the remaining integral, we first transform the sum via the Poisson summation formula.

$$\begin{aligned} \sum_{k \in \mathbb{Z}} [e^y(k+a)]^b e^{-te^{2y} V^2 \frac{1}{4}(k+a)^2} \\ = 2\sqrt{\pi} e^{(b-1)y} V^{-1} t^{-1/2} \sum_{p \in \mathbb{Z}} e^{2\pi i p a} \left(\frac{i}{2\pi} \frac{\partial}{\partial p} \right)^b e^{-t^{-1} 4\pi^2 e^{-2y} V^{-2} p^2}. \end{aligned}$$

In such formulas, we differentiate before evaluating at p integer. Hence,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \int_{e^{-(2+\delta)y}}^{e^{-2y}} t^q [e^y(k+a)]^b e^{-te^{2y} V^2 \frac{1}{4}(k+a)^2} dt \\ = \sum_{p \in \mathbb{Z}} e^{2\pi i p a} \left(\frac{i}{2\pi} \frac{\partial}{\partial p} \right)^b \int_{e^{-(2+\delta)y}}^{e^{-2y}} t^{q-\frac{1}{2}} 2\sqrt{\pi} e^{(b-1)y} V^{-1} e^{-t^{-1} 4\pi^2 e^{-2y} V^{-2} p^2} dt \\ = \sum_{p \in \mathbb{Z}} e^{2\pi i p a} \left(\frac{i}{2\pi} \frac{\partial}{\partial p} \right)^b \int_1^{e^{\delta y}} e^{-(2q+2-b)y} t^{-q-\frac{3}{2}} 2\sqrt{\pi} V^{-1} e^{-t 4\pi^2 V^{-2} p^2} dt. \end{aligned}$$

For b odd, $p = 0$ does not contribute, and this yields

$$\sum_{\mathbf{k} \in \mathbb{Z}} \int_{e^{-(2+\delta)y}}^{e^{-2y}} t^q [e^y(\mathbf{k} + a)]^b e^{-te^{2y}V^2 \frac{1}{4}(\mathbf{k}+a)^2} dt = O\left(e^{-(2q+2-b)y}\right).$$

For b even, $p = 0$ contributes and we have

$$\begin{aligned} \sum_{\mathbf{k} \in \mathbb{Z}} \int_{e^{-(2+\delta)y}}^{e^{-2y}} t^q [e^y(\mathbf{k} + a)]^b e^{-te^{2y}V^2 \frac{1}{4}(\mathbf{k}+a)^2} dt \\ = O\left(e^{-(2q+2-b)y}\right) + O\left(e^{\frac{\delta}{2}y - (2q+2-b)(1+\frac{\delta}{2})y}\right). \end{aligned} \quad (126)$$

□

Lemma 31. *The asymptotic contribution to the index of the terms of weight w in $c^1 e^{-2\pi i b \cdot v} e^{-\frac{i}{2}\tau \mathbf{k}} D e^{2\pi i b \cdot v} e^{\frac{i}{2}\tau \mathbf{k}} \sigma^{-m-j-1} q_{j,B,m} \tilde{u}^B$ is*

$O(e^{(w-3)y})$ for terms of odd \mathbf{k} homogeneity and

$O(e^{(w-3+\delta)y})$ for terms of even \mathbf{k} homogeneity.

Proof. The first c^1 is $c(\nu)$ from our choice of M_y . The trace is zero unless the term has Clifford degree 4. The expression

$$c^1 e^{-2\pi i b \cdot v} e^{-\frac{i}{2}\tau \mathbf{k}} D e^{2\pi i b \cdot v} e^{\frac{i}{2}\tau \mathbf{k}} \sigma^{-m-j-1} q_{j,B,m} \tilde{u}^B,$$

can be expanded as a sum of terms of the form $c(e^I) \sigma^{-m-j-1} v^J e^{ly} \tilde{q}_{jI,Jlm}$ with $\tilde{q}_{jI,Jlm}$ bounded and commuting with Clifford multiplication. This term traces to 0 unless $|I| = 4$. Its weight is then $w \leq 4 + |J| + l - 2m - 2j$. Now we recall our earlier calculations. The contour integral transforms the σ^{-m-j-1} to a t^{m+j} factor. The change of variables $v \rightarrow t^{-1/2}v$ adds an additional factor of $t^{-\frac{1}{2}(|J|+3)}$. Hence this term enters the integrand with a factor $t^{m+j-\frac{1}{2}(|J|+3)} e^{ly}$. Applying Lemma 30, we see that this contributes $O(e^{-(2m+2j-|J|-1-l)y}) = O(e^{(w-3)y})$ for \mathbf{k} odd homogeneity and $O(e^{(w-3+\delta)y})$ for \mathbf{k} even homogeneity. □

In order to apply Lemma 31, we examine the structure of

$$e^{-2\pi i b \cdot v} e^{-\frac{i}{2}\tau \mathbf{k}} D^2 e^{\frac{i}{2}\tau \mathbf{k}} e^{2\pi i b \cdot v} - \sigma$$

in the given frame. First, we compute the coefficients of the Riemannian connection for the metric g' .

In the frame (105) we have

$$\begin{aligned} de^1 &= de^2 = 0, & de^3 &= \cot(\phi) e^2 \wedge e^3, \\ de^4 &= -e^1 \wedge e^4 - d(\ln(V)) \wedge e^4 + e^{-y} V^{-1} *_3 dV. \end{aligned}$$

Hence $c_{ij}^1 = c_{ij}^2 = 0 = c_{1j}^3 = c_{4j}^3$. The only Riemannian connection terms which are not $O(e^{-y})$ are

$$\gamma_{44}^1 = 1 + O(e^{-y}), \quad \gamma_{33}^2 = -\cot(\phi) + O(e^{-2y}) = O(e^{-y/2}). \quad (127)$$

For the others

$$\begin{aligned} \gamma_{23}^4 &= \frac{k}{2}e^{-y} + O(e^{-2y}), & \gamma_{32}^4 &= -\frac{k}{2}e^{-y} + O(e^{-2y}), \\ \gamma_{42}^3 &= -\frac{k}{2}e^{-y} + O(e^{-2y}), & \gamma_{24}^3 &= -\frac{k}{2}e^{-y} + O(e^{-2y}). \end{aligned} \quad (128)$$

Here k denotes the number of centers in TN_k . Elementary computations give

$$\sum_m c_m \Gamma_m^S = -\frac{1}{2}c(e^m \wedge de^m) - c_m \gamma_{ii}^m.$$

In our frame, $\sum_{j=1}^4 e^j \wedge de^j = e^4 \wedge de^4$, and

$$\begin{aligned} e^4 \wedge de^4 &= e^{-y}V^{-1}e^4 \wedge d\omega = e^{-y}V^{-1}e^4 \wedge *_{{\mathbb{R}^3}}dV \\ &= -ke^{-y}V^{-1}e^4 \wedge d\text{Vol}_{S^2} + O(e^{-2y}) = O(e^{-y}). \end{aligned} \quad (129)$$

Also,

$$\sum_i \gamma_{ii}^1 = 1 + O(e^{-y}), \quad \gamma_{ii}^4 = 0 = \gamma_{ii}^3 + O(e^{-2y}). \quad (130)$$

It follows that

$$\sum_m c_m \Gamma_m^S = \frac{k}{2}e^{-y}V^{-1}c(e^4 \wedge d\text{Vol}_{S^2}) - c_1\gamma_{ii}^1 - c_2\gamma_{ii}^2 + O(e^{-2y}). \quad (131)$$

Although the term $\frac{k}{2}e^{-y}V^{-1}c(e^4 \wedge d\text{Vol}_{S^2})$ is $O(e^{-y})$, it is weight one in our filtration, and contributes to our asymptotic computations.

Hence

$$\begin{aligned} D &= c^m \nabla_{e_m} = c^m e_m + c^m \Gamma_m^S + c^m A_m \\ &= c^m e_m + c^m (A_m - \gamma_{ii}^m) + \frac{k}{2}e^{-y}V^{-1}c(e^4 \wedge d\text{Vol}_{S^2}) + O(w \leq -1), \end{aligned}$$

where we write $O(w \leq -k)$ to denote terms of weight $\leq -k$. Therefore,

$$\begin{aligned} e^{-2\pi i b \cdot v} e^{-\frac{i}{2}\tau k} D e^{\frac{i}{2}\tau k} e^{2\pi i b \cdot v} &= c^m (e_m + A_m - \gamma_{ii}^m) + \frac{k}{2}e^{-y}V^{-1}c(e^4 \wedge d\text{Vol}_{S^2}) \\ &- \frac{1}{2}(i\Lambda + 2e^{-y}V^{-1}A_4)(c(\omega) - e^y V c_4) + c(\delta u) + O(w \leq -1) = \hat{D} + c(\delta u). \end{aligned}$$

By Theorem 18

$$\frac{i\Lambda}{2} + e^{-y}V^{-1}A_4 = O(e^{-y}). \quad (132)$$

Therefore, for all m

$$|[\Lambda, A_m]| = O(e^{-y}). \quad (133)$$

Using (132) rewrite \hat{D} as

$$\hat{D} = \sum_{m < 3} c^m(e_m + A_m - \gamma_{ii}^m) + c^4 e_4 + \frac{k}{2} e^{-y} V^{-1} c(e^4 \wedge d\text{Vol}_{S^2}) + O(\mathbf{w} \leq -1). \quad (134)$$

Thus \hat{D} and $c(\delta u)$ have weight 2. Write

$$e^{-2\pi i b \cdot v} e^{-\frac{i}{2} \tau \mathbf{k}} (D^2 - \lambda) e^{\frac{i}{2} \tau \mathbf{k}} e^{2\pi b \cdot v} - \sigma = \hat{D}^2 + \{\hat{D}, c(\delta u)\}.$$

Observe that \hat{D}^2 has weight 2 and $\{\hat{D}, c(\delta u)\}$ has weight 3. From our definition of q_j :

$$q_j = (-1)^j L_j \circ L_{j-1} \circ \cdots \circ L_1 I,$$

where $L_j = \sigma^j(\hat{D}^2 + \{\hat{D}, c(\delta u)\})\sigma^{-j}$. Conjugation by σ^{-j} does not raise weight. Hence, each L_i contributes at most weight 3 to $c^1(\hat{D} + c(\delta u))q_j$. The summand of weight 3 in $\{\hat{D}, c(\delta u)\}$ is

$$c(d\delta u) = -\frac{\cos(\phi)}{\sin^2 \phi} 2\pi i v_3 c(e^2 \wedge e^3) + e^y V \frac{i}{2} (\mathbf{k} - \Lambda) c(e^1 \wedge e^4) + O(\mathbf{w} \leq 2).$$

Observe that

$$c^1(e^y V \frac{i}{2} (\mathbf{k} - \Lambda) c(e^1 \wedge e^4))^j \text{ has weight } j + 1.$$

Thus $e^y V \frac{i}{2} (\mathbf{k} - \Lambda) c(e^1 \wedge e^4)$ should be counted as a weight 1 term, instead of a weight 3 term, when computing its contribution to the weight of $c^1(\hat{D} + c(\delta u))q_j$.

Similarly, observe that when the $-\frac{\cos(\phi)}{\sin^2 \phi} 2\pi i v_3 c(e^2 \wedge e^3)$ term appears an even number of times, the Clifford factors will cancel leaving a term with effective weight 1. If it appears once, the v_3 must be matched with another odd power of v_3 in order to get a non-vanishing integral. Hence, the effective weight of $c(d\delta u)$ is 2, and each L_i raises the weight of $c^1(\hat{D} + c(\delta u))q_j$ by at most 2. Therefore, $c^1(\hat{D} + c(\delta u))\sigma^{-1-j}q_j$ has weight at most 3. By Lemma 31, this is the minimum weight for which we obtain a nonexponentially decreasing contribution to the asymptotic term in the index. Hence, we may discard all terms which have nonmaximal weight or have exponentially decreasing factors multiplied by terms of maximal weight. For example, in our coordinate system, $|\cos(\phi)| = O(e^{-\frac{3}{4}y})$. Hence, the terms $-\frac{\cos(\phi)}{\sin^2 \phi} 2\pi i v_3 c(e^2 \wedge e^3)$ make no contribution.

Lemma 32. *For $j > 1$, the q_j summands vanish in the asymptotic contribution to the index.*

Proof. The terms in $\hat{D} + c(\delta u)$ with Clifford degree less than 2 have weight less than or equal to 2. The terms in L_j with Clifford degree less than 2 are below maximal weight. Since the maximal Clifford degree is 4, the terms in L_j for $j > 1$ are all effectively submaximal, since they must lower Clifford degree. \square

We have

$$c^1(\hat{D} + c(\delta(u))) = \frac{k}{2}e^{-y}V^{-1}c^1c(e^4 \wedge d\text{Vol}_{S^2}) + c^1c(\delta(u)) + O(w \leq 2). \quad (135)$$

The summand $\frac{k}{2}e^{-y}V^{-1}c^1c(e^4 \wedge d\text{Vol}_{S^2})$ has Clifford degree 4; hence, all terms in L_j enter with submaximal weight in $\frac{k}{2}e^{-y}V^{-1}c^1c(e^4 \wedge d\text{Vol}_{S^2})q_j$, $j > 0$. Therefore, this summand only contributes to the q_0 term. On the other hand, $c^1c(\delta(u))q_0$ has Clifford degree < 4 and therefore traces to 0. This reduces our boundary integral (124) to the following.

$$\begin{aligned} & \int_{\partial M_s} \int \text{tr } c^1 \gamma^5 \sum_{k \in \mathbb{Z}} c(\delta u) \frac{-1}{2\pi i} \int_C e^{-t\lambda} \sigma^{-2} q_1 d\lambda dv d\nu \\ & + \int_{\partial M_s} \int \text{tr } c^1 \gamma^5 \sum_{k \in \mathbb{Z}} \frac{-1}{2\pi i} \int_C e^{-t\lambda} \sigma^{-1} \left(\frac{k}{2} e^{-y} V^{-1} c(e^4 \wedge d\text{Vol}_{S^2}) \right) d\lambda dv d\nu + O(e^{-s/2}). \end{aligned}$$

Let us now focus on the q_1 term.

Lemma 33. *The summand $\int \text{tr } c^1 \gamma^5 c(\delta u) \sigma^{-2} q_1 du$ can be replaced by*

$$\int \text{tr } c^1 \gamma^5 \left(-\sigma^{-1} k e^{-y} V^{-1} c(e^2 \wedge e^3 \wedge e^4) + c^4 e^y V \frac{i}{2} (k - \Lambda) \sigma^{-2} q_1 \right) du$$

without altering the asymptotic contribution.

Proof. From the definition of $c(\delta u)$ it follows that

$$c(\delta u) = \frac{i}{4\pi} \left(c^1 \partial_{v_1} + c^2 \partial_{v_2} + \sin \phi c^3 \partial_{v_3} \right) \sigma + c^4 e^y V \frac{i}{2} (k - \Lambda).$$

Therefore,

$$c(\delta u) \sigma^{-2} q_1 = - \left(\frac{i}{4\pi} (c^1 \partial_{v_1} + c^2 \partial_{v_2} + c^3 \sin \phi \partial_{v_3}) \sigma^{-1} \right) q_1 + c^4 e^y V \frac{i}{2} (k - \Lambda) \sigma^{-2} q_1.$$

Integrating by parts,

$$\begin{aligned} \int \text{tr } c^1 \gamma^5 c(\delta u) \sigma^{-2} q_1 du &= \sum_k \int \text{tr } c^1 \gamma^5 \left(\frac{i}{4\pi} \sigma^{-1} (c^1 \partial_{v_1} + c^2 \partial_{v_2} + c^3 \sin \phi \partial_{v_3}) q_1 \right. \\ &\quad \left. + c^4 e^y V \frac{i}{2} (k - \Lambda) \sigma^{-2} q_1 \right) dv. \end{aligned}$$

The $\text{tr } c^1 \gamma^5 \frac{i}{4\pi} \sigma^{-1} c^1 \partial_{v_1} q_1$ summand vanishes in the limit, because its degree 4 Clifford terms are exponentially decreasing. To evaluate the remaining terms recall $q_1 = -\sigma \left(\hat{D}^2 + \{\hat{D}, c(\delta u)\} \right) \sigma^{-1}$. Hence,

$$\partial_{v_j} q_1 = -\sigma \left[\sigma^{-1} \partial_{v_j} \sigma, \hat{D}^2 + \{\hat{D}, c(\delta u)\} \right] \sigma^{-1} - \sigma \frac{1}{(\sin \phi)^{\delta_{j3}}} \{\hat{D}, 2\pi i c^j\} \sigma^{-1}. \quad (136)$$

Using (134), one observes that the only terms of Clifford degree at least 3 in $\sum_{j<4} c^j \{\hat{D}, c^j\}$ are

$$c^2 \{\hat{D}, c^2\} + c^3 \{\hat{D}, c^3\} = -2ke^{-y}V^{-1}c(e^2 \wedge e^3 \wedge e^4) + O(w \leq -1).$$

Next, observe that

$$c^1 [\sigma^{-1} \partial_{v_j} \sigma, \hat{D}^2] = c^1 \left\{ [\sigma^{-1} \partial_{v_j} \sigma, \hat{D}], \hat{D} \right\} = -c^1 \left\{ c(d(\sigma^{-1} \partial_{v_j} \sigma)), \hat{D} \right\}$$

has weight -1 and, therefore, does not contribute to the asymptotic term. \square

Finally, we are left to simplify the term

$$\begin{aligned} & \int \text{tr } c^1 \gamma^5 c^4 e^y V \frac{i}{2} (k - \Lambda) \sigma^{-2} q_1 du \\ &= - \int \text{tr } c^1 \gamma^5 c^4 e^y V \frac{i}{2} (k - \Lambda) \sigma^{-1} \left(\hat{D}^2 + \left\{ \hat{D}, c(\delta u) \right\} \right) \sigma^{-1} du. \end{aligned}$$

The weight 4 term in $\{\hat{D}, c(\delta u)\}$ with $c^1 c^4$ vanishes because it contributes the wrong Clifford factors. The other weight 4 term in $\{\hat{D}, c(\delta u)\}$ is odd in v_3 and vanishes upon dv integration. The remaining terms in $\hat{D}^2 + \{\hat{D}, c(\delta u)\}$, which have Clifford degree 2 and weight 2, are

$$\frac{1}{2} c^i c^j F_{ij}^0 - \frac{i}{2} (k - \Lambda) c(d\omega) = \frac{1}{2} c^i c^j F_{ij}^0 + \frac{ik}{2} (k - \Lambda) c(d\text{Vol}_{S^2}).$$

Here F_{ij}^0 is the zero Fourier mode of F_{ij} . Recall definition (16).

Thus, the asymptotic contribution simplifies to

$$\begin{aligned} & \int_{\partial M_s} \sum_{k \in \mathbb{Z}} \frac{-1}{2\pi i} \int_C e^{-t\lambda} \int \text{tr } c_1 \gamma^5 \left[-\sigma^{-1} k e^{-y} V^{-1} c(e^2 \wedge e^3 \wedge e^4) \right. \\ & \quad - c^4 e^y V \frac{i}{2} (k - \Lambda) \sigma^{-2} \left(\frac{1}{2} c^i c^j F_{ij}^0 + \frac{ik}{2} (k - \Lambda) c(d\text{Vol}_{S^2}) \right) \\ & \quad \left. + \sigma^{-1} \left(\frac{k}{2} e^{-y} V^{-1} c(e^4 \wedge d\text{Vol}_{S^2}) \right) \right] dv d\lambda d\nu + O(e^{-s/2}). \end{aligned}$$

Finally, the only contribution to the asymptotic term of the index comes from

$$4\text{tr}_E \left(\sigma^{-1} \frac{k}{2} e^{-y} V^{-1} - e^y V \frac{k}{4} \sigma^{-2} (k - \Lambda)^2 + e^y V \frac{i}{2} (k - \Lambda) \sigma^{-2} F_{23}^0 \right), \quad (137)$$

where tr_E denotes trace of an endomorphism of E (as opposed to the trace of an endomorphism of $S \otimes E$, where S denotes the spinors).

6.4 The Boundary Contribution

After the reductions obtained in the previous section, the computation of the index reduces to the computation of the sum and integral of (137). The elements $\frac{\lambda_j}{l}$ of the diagonal matrix Λ only enter our computations as $\frac{\lambda_j}{l} + \mathbb{Z}$. We denote by $\{\lambda_j/l\}$ the unique representative of $\frac{\lambda_j}{l} + \mathbb{Z}$ in the interval $[0, 1)$ and we let $\{\Lambda\}$ denote the diagonal matrix with entries $\{\lambda_j/l\}$.

Theorem 34. *The asymptotic contribution to the index equals*

$$\begin{aligned} \lim_{y \rightarrow \infty} \frac{1}{2} \int_{e^{-(2+\delta)y}}^{\infty} \int_{\partial M_y} \text{tr}_E(\nu) \gamma^5 D e^{-tD^2}(x, x) \\ = \frac{k}{2} \text{tr}_E \left(\{\Lambda\}^2 - \{\Lambda\} + \frac{1}{6} \right) + \frac{1}{2} \int_{S_{\infty}^2} \text{tr}_E \left(\frac{\{\Lambda\}}{\pi} i F_{23}^0 - \frac{i F_{23}^0}{2\pi} \right) d\text{Vol}_{S^2}. \end{aligned}$$

Proof. We must evaluate the sums and integrals corresponding to each one of the three summands in (137). We start with the third one

$$\begin{aligned} \sum_{\mathbf{k}} \int_{e^{-(2+\delta)y}}^{\infty} \int_{\mathbb{R}^3} \frac{-1}{2\pi i} \int_C e^{-t\lambda} 4 \text{tr}_E e^{yV} \frac{i}{2} (\mathbf{k} - \Lambda) \sigma^{-2} F_{23}^0 d\lambda dv dt \\ = \sum_{\mathbf{k}} 4 \int_{e^{-(2+\delta)y}}^{\infty} \int_{\mathbb{R}^3} \text{tr}_E e^{-t \frac{e^{2y} V^2}{4} (\mathbf{k} - \Lambda)^2} e^{-t 4\pi^2 (v_1^2 + v_2^2 + \frac{v_3^2}{\sin^2 \phi})} e^{yV} \frac{i}{2} (\mathbf{k} - \Lambda) F_{23}^0 dv dt dt \\ = \sum_{\mathbf{k}} 4 \int_{e^{-\delta y} V^2}^{\infty} \text{tr}_E \sin \phi e^{-\frac{t}{4} (\mathbf{k} - \Lambda)^2} (4\pi)^{-3/2} \frac{i}{2} (\mathbf{k} - \Lambda) F_{23}^0 t^{-1/2} dt. \end{aligned}$$

The Poisson summation formula implies

$$\sum_{\mathbf{k} \in \mathbb{Z}} (\mathbf{k} + a) e^{-4\pi^2 s (\mathbf{k} + a)^2} = \sum_{p=1}^{\infty} 2p \sin(2\pi p a) (4\pi s)^{-3/2} e^{-p^2/4s}.$$

This transforms the last integral to

$$\begin{aligned} -4 \text{tr}_E \sum_{p=1}^{\infty} 2p \sin(2\pi p \Lambda) \sin \phi \int_{e^{-\delta y} V^2}^{\infty} e^{-\frac{4\pi^2 p^2}{t}} \frac{i}{2} F_{23}^0 t^{-2} dt \\ = -4 \text{tr}_E \frac{i}{2} F_{23}^0 \sum_{p=1}^{\infty} 2p \sin(2\pi p \Lambda) \sin \phi \int_0^{e^{\delta y} V^{-2}} e^{-t 4\pi^2 p^2} dt. \end{aligned}$$

In the limit as $y \rightarrow \infty$ this reduces to

$$-\text{tr}_E \sum_{p=1}^{\infty} \frac{\sin(2\pi p \Lambda)}{\pi^2 p} i F_{23}^0 \sin \phi. \quad (138)$$

Recalling the Fourier expansion of the Bernoulli polynomials [EMOT81, Sec.1.13] for $x \in (0, 1)$:

$$\frac{1}{n!} B_n(x) = - \sum_{p \neq 0} \frac{e^{2\pi i p x}}{(2\pi i p)^n}, \quad (139)$$

we have $\frac{1}{2} - \{a\} = \sum_{p=1}^{\infty} \frac{\sin(2\pi pa)}{\pi p}$, where $a \in \mathbb{R} \setminus \mathbb{Z}$ we denote by $\{a\} \in [0, 1)$ the unique representative of $a + \mathbb{Z}$ in that interval. The sum (138) then reduces, under Assumption 22, to

$$\mathrm{tr}_E \left(\frac{\{\Lambda\}}{\pi} - \frac{I}{2\pi} \right) iF_{23}^0 \sin \phi. \quad (140)$$

We recall that in our parametrix construction, we had replaced $d\mathrm{Vol}_{S^2}$ by $d\phi \wedge d\theta$. The factor of $\sin \phi$ in (140) restores the usual volume form, and the contribution of the final summand of (137) to the index is

$$\int_{S_{\infty}^2} \mathrm{tr}_E \left(\frac{\{\Lambda\} iF_{23}^0}{\pi} - \frac{iF_{23}^0}{2\pi} \right) d\mathrm{Vol}_{S^2}$$

Now we consider

$$\begin{aligned} & \sum_{\mathbf{k}} \frac{-1}{2\pi i} \int_{e^{-(2+\delta)y}}^{\infty} \int_C e^{-t\lambda} \int_{\mathbb{R}^3} \mathrm{tr}_E (\sigma^{-1} \frac{k}{2} e^{-y} V^{-1} - e^y V \frac{k}{4} \sigma^{-2} (\mathbf{k} - \Lambda)^2) dv d\lambda dt = \\ & \sum_{\mathbf{k}} \int_{e^{-(2+\delta)y}}^{\infty} \int_{\mathbb{R}^3} \mathrm{tr}_E e^{-t \frac{2yV^2}{4} (\mathbf{k} - \Lambda)^2} e^{-t4\pi^2 (v_1^2 + v_2^2 + \frac{v_3^2}{\sin^2 \phi})} (\frac{k}{2} e^{-y} V^{-1} - t e^y V \frac{k}{4} (\mathbf{k} - \Lambda)^2) dv dt = \\ & \sum_{\mathbf{k}} \int_{e^{-\delta y} V^2}^{\infty} \mathrm{tr}_E e^{-\frac{t}{4} (\mathbf{k} - \Lambda)^2} (4\pi t)^{-\frac{3}{2}} (\frac{k}{2} - t \frac{k}{4} (\mathbf{k} - \Lambda)^2) \sin(\phi) dt. \end{aligned}$$

The Poisson summation formula implies

$$\begin{aligned} \sum_{\mathbf{k} \in \mathbb{Z}} e^{-4\pi^2 s (\mathbf{k} + a)^2} &= \sum_{p \in \mathbb{Z}} (4\pi s)^{-1/2} e^{-p^2/4s} e^{2\pi i p a}, \text{ and} \\ \sum_{\mathbf{k} \in \mathbb{Z}} e^{-4\pi^2 s (\mathbf{k} + a)^2} 4\pi^2 s (\mathbf{k} + a)^2 &= \sum_{p \in \mathbb{Z}} (4\pi s)^{-1/2} e^{-p^2/4s} e^{2\pi i p a} \left(\frac{1}{2} - \frac{p^2}{4s} \right). \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{\mathbf{k}} \int_{e^{-\delta y} V^2}^{\infty} \mathrm{tr}_E e^{-\frac{t}{4} (\mathbf{k} - \Lambda)^2} (4\pi t)^{-\frac{3}{2}} (\frac{k}{2} - t \frac{k}{4} (\mathbf{k} - \Lambda)^2) \sin(\phi) dt \\ &= \frac{k}{2} \sum_p \int_{e^{-\delta y} V^2}^{\infty} \mathrm{tr}_E e^{-\frac{4\pi^2 p^2}{t}} e^{-2\pi i p \Lambda} \frac{2\pi p^2}{t} \sin(\phi) t^{-2} dt \\ &= \frac{k}{2} \sum_p \int_0^{e^{\delta y} V^{-2}} \mathrm{tr}_E e^{-4\pi^2 p^2 t} e^{-2\pi i p \Lambda} 2\pi p^2 \sin(\phi) t dt \\ &= \frac{k}{8\pi} \sin(\phi) \mathrm{tr}_E \sum_{p>0} \frac{\cos(2\pi i p \Lambda)}{\pi^2 p^2} + O(e^{-4e^{\delta y} V^{-2} \pi^2}). \end{aligned}$$

The Bernoulli polynomial Taylor expansion (139),

$$\sum_{p>0} \frac{\cos(2\pi pa)}{\pi^2 p^2} = B_2(a) = \{a\}^2 - \{a\} + \frac{1}{6},$$

reduces this defect term to

$$-\mathrm{tr}_E *_3 dV(\hat{e}_2, \hat{e}_3) \sin \phi \frac{\{\Lambda\}^2 - \{\Lambda\} + \frac{I}{6}}{4\pi}.$$

□

Assembling the above results we obtain

$$\begin{aligned} \mathrm{ind}_{L^2} D^+ &= \frac{k}{2} \mathrm{tr}(\{\Lambda\}^2 - \{\Lambda\}) + \frac{1}{2} \mathrm{tr} \left(\int_{S_\infty^2} \frac{\{\Lambda\}}{\pi} iF_{23}^0 - \int_{S_\infty^2} \frac{iF_{23}^0}{2\pi} \right) d\mathrm{Vol}_{S^2}. \\ &\quad + \frac{1}{8\pi^2} \int \mathrm{tr} F \wedge F. \end{aligned} \quad (141)$$

From the asymptotic form of the connection of Theorem 18 we easily evaluate the boundary contribution, given by the first line of (141). Letting $M = \mathrm{diag}(m_j)$, we have $\frac{i}{2\pi} \int_{S_\infty^2} F_{23} d\mathrm{Vol}_{S^2} = \frac{1}{2\pi} \int_{S^2} d \frac{\Lambda(d\tau + k\nu) - M\nu}{2} = M - k\Lambda$. Thus we obtain

Theorem 35. *The index of the Dirac operator D_A equals*

$$\begin{aligned} \mathrm{ind}_{L^2} D^+ &= \mathrm{tr} \left(\frac{k}{2} \{\Lambda\}^2 - \frac{k}{2} \{\Lambda\} - \{\Lambda\}(k\Lambda - M) + \frac{1}{2}(k\Lambda - M) \right) \\ &\quad + \frac{1}{8\pi^2} \int \mathrm{tr} F \wedge F. \end{aligned} \quad (142)$$

Let's apply this formula to the basic instanton on TN_1 . It is given by $A = -i\frac{s}{2V}(d\tau + \omega)$, with curvature $F = -i(d(s/2V) \wedge (d\tau + \omega) + (s/2V)d\omega)$. Therefore, $\Lambda = s/l$, $M = 0$, and $\frac{1}{8\pi^2} \int F \wedge F = \frac{1}{2}(s/l)^2$. The above index formula reduces to $[s/l]([s/l] + 1)/2$, is in agreement with [Pop81], where the solutions of the Dirac operator on TN_1 in this background were explicitly found. These solutions were recently studied in [JS14, JS16].

As another illustration, consider a Whitney sum $\oplus_{j=1}^n L_j$ of line bundles with abelian connection $-ia_j = \frac{i}{2} \left(\frac{H_j}{V}(d\tau + \omega) - \pi_k^*(\eta_j) \right)$ on L_j , with $H_j = \lambda_j + \sum_\sigma \frac{v_{j\sigma}}{r_\sigma}$ and $d\eta_j = *_3 dH_j$. Thus, we have an instanton $A = -i\mathrm{diag}(a_j)$. Its second Chern character value is $\frac{1}{8\pi^2} \int F_A \wedge F_A = \frac{1}{2} (k\Lambda^2 - 2\Lambda M + \mathrm{diag}(\sum_\sigma (v_{j\sigma})^2))$, giving

$$\mathrm{ind}_{L^2} D^+ = \sum_{j=1}^n \sum_{\sigma=1}^k \frac{([\lambda_j/l] - v_{j\sigma})([\lambda_j/l] - v_{j\sigma} + 1)}{2}. \quad (143)$$

Acknowledgements

SCh is grateful to the Institute for Advanced Study, Princeton and to the Institute des Hautes Études Scientifiques for their hospitality and support during various stages of this work; he also thanks the Berkeley Center for Theoretical Physics for hospitality during its completion. The work of SCh was partially supported by the Simons Foundation grant #245643. The work of MAS was partially supported by the Simons Foundation grant #353857 and NSF grant DMS 1005761. The work of ALH was supported by an NSF Alliance Postdoctoral Fellowship.

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